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This report describes accomplishments in developing methods of system identification for robust control design. The starting point is an *a priori* plant description containing both parametric and nonparametric uncertainty. The identification methods are developed under differing *a priori* assumptions on the parametric and nonparametric parts of the model set. For example, when a bound on the nonparametric part is known, it is shown that the parameters in the parametric part of the model are contained in either an ellipsoid or hyperboloid, depending on the data. Computational methods are very similar to standard least-squares methods and can be computed in a batch or recursive manner. The parameter set membership description is used for robust control design via a mini-max optimization problem. Other approaches explored include high-order ARX models which produce purely parametric uncertainty under standard statistical assumptions on the disturbances. A learning scheme is also investigated where the control and identification are iteratively coupled by the closed-loop.

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Set-Membership Identification for Robust Control Design

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Prepared by:

Dr. Robert L. Kosut

Integrated Systems Inc.
3260 Jay St.
Santa Clara, California 95054

Prepared for:

Air Force Office of Scientific Research (AFOSR)
Directorate of Mathematical and Computer Sciences
Building 410
Bolling Air Force Base
Washington, DC 20332

Approved by:

R.L. Kosut
.....
Dr. Robert L. Kosut
Manager, Research Group

R.M. Dressler
.....
Dr. Robert M. Dressler
Vice President, Adavanced Systems Department

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Integrated Systems, Inc.
3260 Jay Street
Santa Clara, CA 95054-3309
Tel: 408.980.1500 Fax: 408.980.0400

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This report describes work performed from 1 Sept. 89 through 31 March 93 under funding from AFOSR, Directorate of Mathematical and Computer Sciences, Contract No. F49620-89-C-0119. The Principal Investigator was Dr. Robert L. Kosut, Manager of the Research Group at Integrated Systems, Inc. Prof. Stephen Boyd from the Department of Electrical Engineering, Stanford University was a Consultant. Ming Lau, while a Ph.D. candidate at Stanford under the guidance of Prof. Gene Franklin and Prof. Boyd, contributed significantly to the research in ellipsoid set estimation and robust control design. The work reported here on iterative control and identification is the result of collaboration between Dr. Kosut and Prof. Brian D. O. Anderson from the Department of Systems Engineering, The Australian National University.

Chapter 1

Introduction

1.1 Background and Motivation

Current approaches to robust control take for granted the availability of uncertainty descriptions, *e.g.*, parameters lying in fixed intervals (*e.g.*, Barmish[5], Biernacki *et al.*[9]) or frequency domain ($\mathcal{H}_\infty$) bounds (*e.g.*, Safonov *et al.*[60], Doyle *et al.*[16], Francis and Zames[22]). However, the question remains as to how these descriptions might be obtained in practice. On the other hand, the identification community has emphasized estimation of nominal models without developing an associated estimate of model quality. When model error evaluation has been carried out, this usually accounted only for random effects due to exogenous inputs rather than errors due to inherent model limitations which necessitate a robust control design, *e.g.*, Jenkins and Watts[31], Ljung[49] and the references therein. There is now a greater recognition by both communities of the requirements of the other. This recognition is evidenced by the strong interest shown by researchers from both the identification and robust control communities, *e.g.*, the recent Special Issue [35], and the many sessions on this topic at recent conferences and workshops.

Despite this research activity, this subject is still in its infancy and many developments are likely to arise from intensive research efforts devoted to the interaction between the previously separate fields of identification and robust control. To fill the needs of robust control design will require a new approach to system identification which provides both a nominal model and a measure of its uncertainty. We refer to this approach as "set-membership identification" or "set estimation."

The long-range goal of this research is to form a new system identification paradigm that fulfills all the requirements of robust control design. This will have a significant impact in the engineering community where such an "engineering theory" is badly needed. Moreover, with the wide availability and use of CACSD packages, such as MATRIX$_X$ research results will be rapidly spread. Since system identification and robust control design are ubiquitous engineering activities, the benefits of this research will be widely utilized, particularly among control engineers involved with aircraft, spacecraft, robotics, and industrial automation.

This report documents our research efforts which concentrated almost exclusively on set-estimation. Some effort was spent on the important next step of robust controller design using the estimated model accuracy.

In the remainder of this chapter we provide an overview of the issues and a brief summary of our results.

1.2 Model Accuracy Estimation

As expounded by Ljung[49], identification consists of three essential ingredients, namely, (i) measured data, (ii) a candidate model set, and (iii) a criteria for selecting a candidate model using the data. Moreover, all three should be selected based on the intended use of the identified model. The problem is the model set which traditionally consists of a *single* parametric model. There is no associated parametrization in the model set of a measure of uncertainty. Thus, the designer must guess or have faith in the identified model when used for controller design. But this opposes all the standing assumptions made in current robust control design methods. These methods require a *set of models*, not a single model. For example, a model set can consist of a transfer function which depends in a known way on uncertain parameters, or the set may be described as a nominal model together with a frequency dependent "ball of uncertainty".

The integration of control design and identification is not altogether a new issue. The most familiar and appealing application is *adaptive control* where, as shown in figure 1.1, a model is identified concurrently with the on-line optimization of the control law based on the model. This leads to intricate nonlinear recursions which have not been fully understood to date. There are global stabilization schemes which are not robust; there are local stability results applicable to the steady-state, and hardly anything is known about the transient behavior of adaptive systems, *e.g.*, Åström and Wittenmark[4], Anderson *et al.*[2].

A formulation where explicit control action is anticipated for the purpose of identification is the so called "dual control" design, *e.g.*, Feldbaum[19], Barshalom and Tse[6]. Due to the high computational requirements associated with this method, implementation is only possible with crude approximations which lead to similar problems as with the adaptive case.

The approach we have been pursuing, illustrated in figure 1.2, is a two step procedure, where identification produces a nominal model along with an uncertainty profile. The control is then designed to be robust with respect to the estimated model set. This results in an iterative solution where models and control are adapted to the changing experimental conditions. This differs considerably from the classical adaptive control scheme (figure 1.1) where the estimator produces a single model with no information about model accuracy. In the robust control design procedure of the new approach (figure 1.2), the plant model is replaced by a model set which reflects the accuracy with which the model has been estimated.

In the work described here, we formulate a model set and an identification criterion from which set-membership identification that uses time-domain data and meets the requirements of robust control design, naturally follows. Specifically, we have investigated the following topics:

1. high order least-squares set-estimation with ARX model sets.
2. robust control with uncertain ARX model sets.
3. ellipsoid sets with known nonparametric uncertainty.
4. robust control of ellipsoid sets.
5. $\mathcal{L}_\infty$ identification.

Before we discuss the results of our efforts, there are some other relevant issues to clarify. Specifically, the character of uncertainty, computation, and MIMO systems.

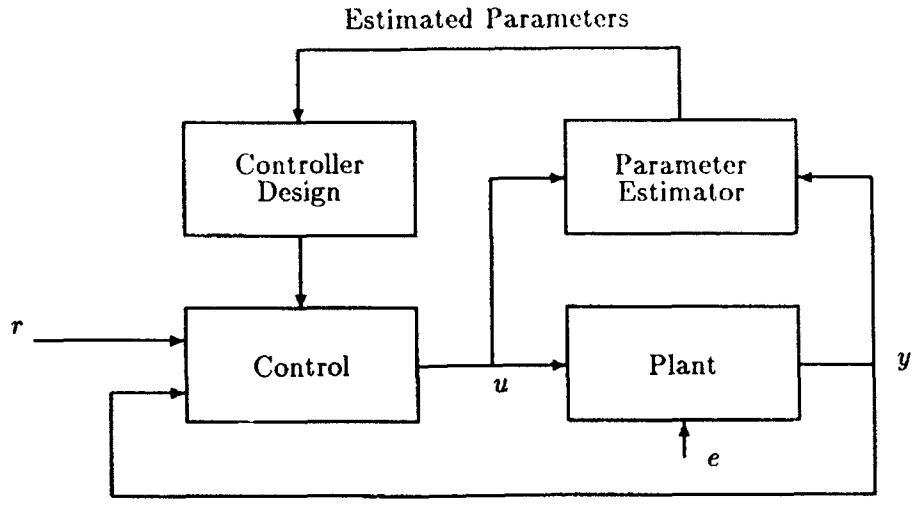

Figure 1.1: Traditional adaptive control system with parameter estimator.

1.3 On the Character of Uncertainty

The current debate amongst researchers involved with set-membership identification centers on the nature of the set itself: is it probabilistic or deterministic/worst-case. Clearly both can be used to quantify uncertainty in either disturbances and transfer functions. A probabilistic, or stochastic, description of a disturbance is common practice and forms the basis for $\mathcal{H}_2$-filtering and control design, *i.e.*, optimal filtering and LQG control design. A power bounded set of disturbances and/or a deterministic/worst-case description of transfer function uncertainty leads to $\mathcal{H}_\infty$ methods of control design, *e.g.*, Doyle *et al.*[15]. These sets can be combined leading to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design, *e.g.*, Khargonnekar and Rotea[33].

If we begin with a stochastic description of the exogenous inputs to a system, then the high-order least-squares based identification methods described in section 2.2.2 lead naturally to the use of a probabilistic set to describe the dynamic uncertainty, which is purely parametric. This result immediately raises the question: what does a robust control mean in the context of probabilities? We tend to think of a robust controller as providing an absolute guaranty against instability and/or certain levels of performance degradation given a deterministic, or "hard " bound on plant uncertainty. With a probabilistic description, or "soft" bound, we must decide if 99.99% is safe enough. To turn the question the other way, the deterministic bounds necessitates guarding against the worst-case. But conditions for the worst-case may be extreme, thereby leading to an overly conservative controller. But this brings us back to exactly the question of probabilities and outcomes, and finally to a more fundamental question: is Nature neutral or conspiratorial?

Attempting an answer at this time may not be necessary, nor very fruitful. Our philosophy has been more pragmatic. We will leave it be, and follow the consequences of different assumptions by developing a theory of set-membership identification and corresponding (as necessary) "robust" control design methods compatible with both probabilistic and deterministic plant sets. In this way we can explore without prejudice.

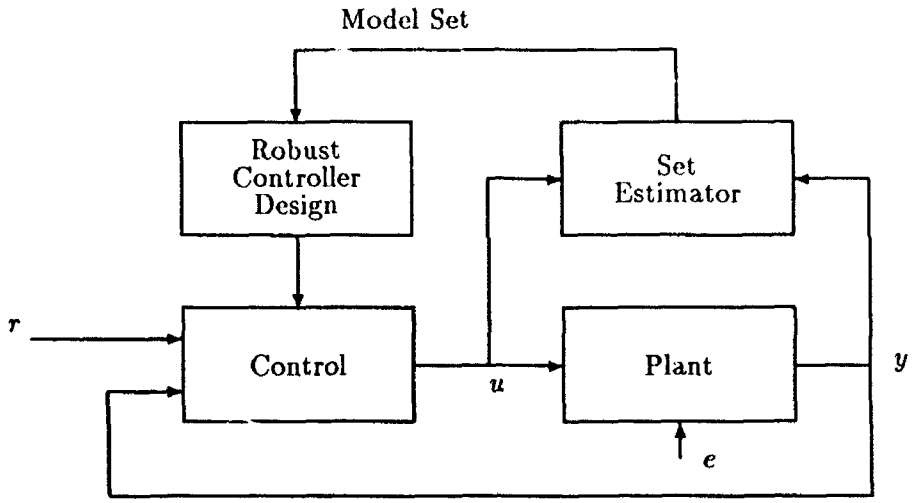

Figure 1.2: Adaptive control with set estimator.

1.4 Computing the Estimate

The computational issue is very relevant to system identification. The great appeal of "least-squares," and the principle reasons for its ubiquity, are because a unique minimum is always obtained, and there are very efficient and reliable methods for computing the solution. The computational methods typically involve square-root algorithms such as the QR transformation, SVD algorithms, as well as lattice forms for very high model orders. It is imperative that the calculations are done in this manner, for otherwise significant numerical errors will accrue, even for a small number of parameters. There are other reasons as well for using a QR method, *e.g.*, (1) high model orders and large amounts of data are easily handled, (2) data from different experiments are readily combined without re-doing the entire estimation, and (3) prediction errors can be computed for varying model orders directly from the QR transformation. These factors make it possible to easily and rapidly generate extremely high order models from large amounts of data. A least squares approach to set estimation will naturally benefit from all the existing computational theory and software.

1.5 MIMO Extensions

All the methods discussed have their MIMO extensions. The arguments made for using high-order ARX models of SISO systems apply equally well to MIMO systems. Similarly, the Toeplitz based methods are also extensible to MIMO systems. So, in principal the methodologies should carry forward. However, issues of parametrization can become very important because with too many inputs and outputs, the number crunching can get out of hand. Unfortunately, extending the parametric robustness tests to the multivariable case is not solved.

1.6 Brief Summary and Relation to Other Approaches

It is fair to say that many of the ideas discussed here for set-estimation have been influenced by the efforts of other researchers as well as by our own previous success and failures. In what follows we give a brief summary of some of the recent literature.

1.6.1 Ellipsoid Parameter Bounds

In our previous work on set-membership identification, we showed how to obtain a set of models that are consistent with a given set of data and a given set of prior assumptions on the possible nonparametric uncertainty and disturbances see section 2.3 for a brief discussion; more details are in Kosut *et al.*[36, 39, 47, 40] and the papers in the Appendix). In these papers the model parameters are shown to lie in a set defined by a quadratic form, i.e., an ellipsoid or hyperboloid, depending on the data A similar approach was used in Younce and Rohrs[70], Wahlberg and Ljung[65]. Earlier versions of this approach based on least-squares are in Kosut[40] and the related robust control of ellipsoid bounded plants in Lau *et al.*[45, 46, 44]. In Wahlberg[65], Laguerre expansions were used to model the dominant system dynamics and ellipsoid bounds also obtained.

The difficulty with the above approaches is that in order to compute the ellipsoid bound, a hard bound on the non-parametric dynamics is required, which unfortunately, is precisely the knowledge which may be difficult to obtain. Another important point is that these methods are based on sufficient conditions to satisfy the prior $\mathcal{H}_\infty$ bound, hence, the sets can be conservative. In the recent work of Poola *et al.*[57], both necessary and sufficient conditions are established, but these are used for model validation.

1.6.2 Stochastic Embedding

In Goodwin *et al.*[25, 26, 24] a stochastic embedding philosophy is adopted (see section 2.3.3 for a brief description). It is assumed that both the unmodelled dynamics and noise are drawn from a probabilistic set having certain amplitude and smoothness properties. These properties are then estimated by maximum likelihood techniques resulting in what we have called here, a probabilistic set-membership estimator. These ideas have motivated our method of using high-order least-squares to estimate the set. The use of high-order least squares as discussed here is also discussed in Kosut and Anderson[37].

1.6.3 Model Order Reduction

The use of Laguerre expansions, as mentioned above, may prove very useful in our high-order least-squares approach, because the orders can be significantly reduced prior to LS estimation, *e.g.*, Wahlberg[66, 67].

1.6.4 Iterative Identification and Control Design

Several approaches have been put forward which involve iterating on closed-loop data while successively adjusting data filters for identification and redesigning the controller, *e.g.*, Schrama[59],Lee

et al.[48], Zang *et al.*[71], Hansen *et al.*[29], Yam *et al.*[69], Kosut[41]. The techniques discussed here for set-membership identification are a necessary part of these schemes.

1.6.5 $\mathcal{H}_\infty$ Identification

Several researchers have considered the problem of identification using the $\mathcal{H}_\infty$ norm starting from bounded error frequency response data at a finite set of frequencies, *e.g.*, Parker and Bitmead[56], Gu and Khargonnekar[32], Helmicki *et al.*[30]. Both linear and nonlinear algorithms have been developed and bounds on the worst-case identification error are also derived. Although there are some very interesting results contained in this work, we would rather start from time-domain data, which really is the source of frequency domain data in the first place. The new methods of $\mathcal{L}_\infty$ identification described in 2.4 and Massoumnia and Kosut[51] (see Appendix) may prove to be a more direct approach to this problem.

1.6.6 Set-Membership Validation

A related problem to set-membership identification is that of model set validation. In Smith and Doyle[63, 62], the following model validation question is posed: "Given experimental data and a model with both additive noise and norm bounded perturbations, is it possible that the model could produce the observed input-output data?" This question is a first step towards the reconciliation of prior assumptions on disturbance and model accuracy with observed data from a system. The approach is based on frequency domain data with a μ-like model structure.

In Poola *et al.*[57], the model validation problem is posed using time-domain data and both necessary and sufficient conditions are obtained for model validation, and hence, invalidation. Our previous work on set-membership identification used only the sufficient conditions to produce the ellipsoidal sets. The underlying theory in Poola *et al.*[57], which provides both the necessary and sufficient conditions for consistency, is based on certain Toeplitz forms. There are some similarities with the Toeplitz forms used the new $\mathcal{L}_\infty$ identification methods discussed in section 2.4.

Chapter 2

Set-Membership Identification

In this section we give an overview of the fundamental problem of set estimation and a detailed summary of our own contributions. The complete details of our work is contained in several papers which are included in this report as an Appendix.

2.1 Problem Formulation

To illustrate the issues, suppose that the true, but unknown system to be controlled is the single-input-single-output stable *discrete-time* system,

$$\mathbf{S} : \{y = Gu + He \mid e \in \mathbf{E}(\lambda)\} \tag{2.1}$$

where G and H are unknown causal linear-time-invariant (LTI) systems with transfer functions $G(z)$ and $H(z)$, respectively. The sequences y and u are, respectively, the sensed output, and the applied control input. The sequence e is unpredictable except known to be in a set $\mathbf{E}(\lambda)$ where λ is unknown. Likely candidates for $\mathbf{E}(\lambda)$ are $\mathbf{E}_{pow}(\lambda)$, the set of sequences with power bound λ, or $\mathbf{E}_{iid}(\lambda)$, iid zero-mean sequences with variance λ. For robust control design, it is necessary to have a set description of the plant system. For example, consider the set[1]

$$\mathbf{M} : \left\{y = (\widehat{G} + \Delta\widehat{W})u + \widehat{H}e \;\middle|\; \|\Delta\|_{\mathcal{H}_\infty} \le \widehat{\gamma},\ e \in \mathbf{E}(\widehat{\lambda})\right\}$$

If $\mathbf{E}(\widehat{\lambda}) = \mathbf{E}_{pow}(\widehat{\lambda})$, then $\mathbf{M}$ is typical for $\mathcal{H}_\infty$ control design. If $\mathbf{E}(\widehat{\lambda}) = \mathbf{E}_{iid}(\widehat{\lambda})$, then mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design methods apply. There are many combinations possible. However, in all the above cases, the quantities with "hats" are available *a priori* to the designer. The problem addressed here, referred to as *set estimation*, is to determine these quantities *a posteriori* from the finite data record,

$$\{y_t, u_t \mid t = 1, \ldots, N\}$$

where y_t and u_t are the values of the sequences y and u, respectively, at time t. In the remainder of this section, some of the issues involved in set estimation are discussed and some promising methods recently proposed are reviewed. More details on these specific techniques can be found in the special issue [35] and the references therein.

[1] If Δ is stable, $\|\Delta\|_{\mathcal{H}_\infty} = \sup_\omega |\Delta(e^{j\omega})|$, otherwise, $\|\Delta\|_{\mathcal{H}_\infty} = \infty$.

2.2 Least-Squares Parameter Estimation

Least-squares (LS) methods of parameter estimation enjoy a very wide usage, and the underlying theory is well developed, especially in a probabilistic framework. In section 2.2.2 we show that the LS estimator together with high-order ARX models lead naturally to transfer function uncertainty which is parametric. Moreover, the parameter uncertainty can be either probabilistic or deterministic, depending on prior assumptions.

Parametric uncertainty has proven much more difficult for robust control design than the non-parametric dynamic uncertainty associated with $\mathcal{H}_\infty$ methods. However, as discussed here, the parametric uncertainty set produced by high-order least-squares seems to be quite tractable and leads to some new approaches to robust control design (section 2.2.3).

The high-order ARX model sets, although compatible with the assumptions in the LS theory, can be viewed as an intermediate step to encoding the data into a model more suitable for robust control design. To reduce the model order, we have examined the use of Laguerre expansions (section 2.2.4) before LS is applied. The selection of the Laguerre kernels may have to be based on *a priori* information, or depend on a desired closed-loop bandwidth.

2.2.1 Statistical Analysis

Parameter estimation via least-squares with an ARX model is perhaps the most widely used approach to system identification. Consider the parametric ARX model set:

$$\mathbf{M} : \{A_\theta y = B_\theta u + e \mid \theta \in \mathbb{R}^p,\ e \in \mathbf{E}_{iid}(\lambda)\} \tag{2.2}$$

where

$$\begin{aligned} A_\theta &= 1 + \sum_{i=1}^{n} a_i z^{-i}, \quad B_\theta = \sum_{i=1}^{m} b_i z^{-i} \\ \theta &= [a_1 \cdots a_n\ b_1 \cdots b_m]^T \end{aligned}$$

Thus,

$$\begin{aligned} y_t &= \phi_t^T \theta + e_t \\ \phi_t^T &= [-y_{t-1} \cdots - y_{t-n}\ u_{t-1} \cdots u_{t-m}] \end{aligned}$$

The least-squares parameter estimate, based on a finite data record, is found from:

$$\widehat{\theta} = \arg\min_\theta \frac{1}{N} \sum_{t=1}^{N} (y_t - \theta^T \phi_t)^2 \tag{2.3}$$

It is well known (Ljung[49]) that

$$\widehat{\theta} \to \theta^*, \text{ as } N \to \infty, \text{ w.p. 1}$$

where

$$\theta^* = \arg\min_\theta \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{err}(\omega, \theta) d\omega$$

with the "error" spectrum given by,

$$\begin{aligned} S_{err}(\omega,\theta) &= |A_\theta(e^{j\omega})G(e^{j\omega}) - B_\theta(e^{j\omega})|^2 S_{uu}(\omega) \\ &+ \lambda |A_\theta(e^{j\omega})H(e^{j\omega})|^2 \end{aligned}$$

In addition, *if the system* (2.1) *is in the ARX model set* (2.2), then the parameter error $\hat{\theta} - \theta^*$ is asymptotically normally distributed, *i.e.*, as $N \to \infty$,

$$\sqrt{N}(\hat{\theta} - \theta^*) \to \mathcal{N}(0, \lambda \mathcal{E}(\phi_t \phi_t^T)^{-1}) \tag{2.4}$$

where $\mathcal{E}(\cdot)$ denotes expectation. Observe that the system (2.1) is in the ARX model set if there exists a parameter θ_0 such that, $G = B_{\theta_0}/A_{\theta_0}$ and $H = 1/A_{\theta_0}$. Although this is not true in general, the true system *can* be arbitrarily well approximated by a high order ARX model. Specifically, set $n = m$. Then, there is a sufficiently large value of n and a corresponding parameter $\theta_0 \in \mathbb{R}^{2n}$ such that $\|H^{-1}G - B_{\theta_0}\|_{\mathcal{H}_\infty}$ and $\|H^{-1} - A_{\theta_0}\|_{\mathcal{H}_\infty}$ are arbitrarily small. Hence, for some sufficiently large values of N and n, reasonable estimates of $\mathcal{E}(\phi_t \phi_t^T)$ and λ, are

$$\hat{R} = \frac{1}{N}\sum_{t=1}^{N} \phi_t \phi_t^T \quad \hat{\lambda} = \frac{1}{N-2n}\sum_{t=1}^{N} \hat{e}_t^2$$

with

$$\hat{e}_t = y_t - \phi_t^T \hat{\theta}$$

the estimated prediction error. The above asymptotic approximations inspire several types of *high-order* ARX set estimators.

2.2.2 High-Order ARX Sets

Let $\hat{G} = \hat{B}/\hat{A}$ and $\hat{H} = 1/\hat{A}$ denote LS/ARX estimates of G and H. Let $m = n$ where n is large. Then, the true system is well approximated by,

$$\hat{A}y = \hat{B}u - \delta^T \hat{R}^{-1/2}\phi + e \tag{2.5}$$

where $\delta \in \mathbb{R}^{2n}$ is the normalized (unknown) parameter error:

$$\delta = \hat{R}^{1/2}(\hat{\theta} - \theta) \tag{2.6}$$

Since $e \in \mathbf{E}_{iid}(\lambda)$, for large N, we have the following approximate statistical properties:

$$\delta \in \mathcal{N}(0, \frac{\lambda}{N} I_{2n}), \quad (N-2n)\frac{\hat{\lambda}}{\lambda} \in \chi^2(N-2n)$$

Therefore,

$$\frac{N}{\lambda} \in \chi^2(p), \quad \frac{\frac{N}{2n}\delta^T\delta}{\hat{\lambda}} \in F(2n, N-2n)$$

where $F(2n, N-2n)$ is the F-distribution with degrees of freedom $2n$ and $N-2n$. Hence,

$$\text{Prob}\{\delta^T\delta \leq \frac{2n}{N}\alpha\hat{\lambda}\} = \eta$$

can be determined from an F-distribution table. To be safe, suppose we set η very high, say, $\eta = .999$. Then for typical numbers such as $N \geq 1000$ and $n = 10$, we get $\alpha = 2.27$. For large n,

say $n = 60$, and large $N \gg n$, we get $\alpha \approx 1.45$, and so on. In addition, for large N, $e \in \mathbf{E}(\hat{\lambda})$. Hence, for large n and large N, the system (2.1) is in the model set

$$\mathbf{M}_{arx} : \begin{cases} \hat{A}y = \hat{B}u - \delta^T \hat{R}^{-1/2}\phi + e \\ \delta^T\delta \leq \frac{2n}{N}\alpha\hat{\lambda} \\ e \in \mathbf{E}_{iid}(\hat{\lambda}) \end{cases} \tag{2.7}$$

with probability of at least η.

It is interesting to compare the above probabilistic result with a strictly deterministic view. For example, the orthogonality properties of the least-squares estimator give:

$$\delta^T\delta = \sum_{t=1}^{N} e_t^2 - (N-2n)\hat{\lambda}$$

This property *requires no probabilistic assumptions on the data.* Hence,

$$\frac{1}{N}\sum_{t=1}^{N} e_t^2 \leq \eta \Rightarrow \delta^T\delta \leq N(\eta - \hat{\lambda}) + 2n\hat{\lambda}$$

The estimate $\hat{\lambda}$ is a possible choice for η which gives a result very similar to that above.

2.2.3 Robust Control with ARX Sets

In this section we discuss the issue of robust control design under the assumption that the true system is in the ARX model set $\mathbf{M}_{arx}$ of (2.7). Suppose we apply the LTI feedback controller

$$u = -\widehat{K}y \tag{2.8}$$

where $\widehat{K}$ stabilizes the "nominal" ARX system ($\delta = 0$),

$$\hat{A}y = \hat{B}u + e$$

Applying the control to the actual system model (2.5), gives the closed-loop system

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} T_\delta e \\ -Q_\delta e \end{bmatrix} = \frac{1}{1-\delta^T h}\begin{bmatrix} \hat{T}e \\ -\hat{Q}e \end{bmatrix}$$

where

$$\hat{T} = \frac{1}{\hat{A} + \hat{B}\widehat{K}}, \quad \hat{Q} = \frac{\widehat{K}}{\hat{A} + \hat{B}\widehat{K}}$$

$$\hat{h} = \hat{R}^{-1/2}\begin{bmatrix} D\hat{T} \\ D\hat{Q} \end{bmatrix}, \quad D = \begin{bmatrix} z^{-1} \\ \vdots \\ z^{-n} \end{bmatrix}$$

Because $\widehat{K}$ stabilizes the nominal system, $\hat{T}$, $\hat{Q}$ and $\hat{h}$ are all stable.

Recall from the Nyquist theorem that since $\hat{h}$ is stable, the closed-loop system is stable if and only if,

$$|1 - \delta^T\hat{h}(e^{j\omega})| \neq 0, \forall \delta^T\delta \leq \rho^2, \forall\omega$$

This is equivalent to

$$\rho < \rho_{\text{stab}}$$

where ρ_{stab}, the so-called "real" stability margin is given by,

$$\rho_{\text{stab}}^2 = \inf_{\omega} \inf_{r(\omega)} \left\{ \delta^T \delta = r(\omega) \;\middle|\; \delta^T \hat{h}(e^{j\omega}) = -1 \right\}$$

Calculating $r(\omega)$ involves finding the minimum norm (least-squares) solution to the over-determined set of equations $\delta^T \hat{h}(e^{j\omega}) = 1$ at each frequency. Thus,

$$r(\omega) = \begin{cases} 1/[\|a\|^2 - (a^T b)^2 / \|b\|^2], & b \neq 0 \\ 1/\|a\|^2, & b = 0 \end{cases}$$

where

$$a = \text{Re } h(e^{j\omega}), \quad b = \text{Im } h(e^{j\omega})$$

Hence, a "probability of stability" can be stated as follows. If

$$\text{Prob}\{\delta^T \delta \leq \rho^2\} = \eta$$

then

$$\rho < \rho_{\text{stab}} \Rightarrow \text{Prob}\{(1 - \delta^T h)^{-1} \text{ stable}\} \geq \eta$$

It ought to be mentioned that no closed form solution is known for the stability margin, ρ_{stab}, in the MIMO case.

2.2.4 Order Reduction via Laguerre Expansions

Although high-order ARX model set estimation seems promising, there are some obvious impediments. First, the controller (2.8) will also be of high order. Secondly, a determination of what is meant precisely by high order is dependent on *a priori* knowledge about the true system. Thirdly, the statistical properties are based on very large data lengths, and again, a precise value depends on the true system properties.

To offset the high order, an alternative is to use a more parsimonious model parametrization. For example, using Laguerre expansions, as proposed in Wahlberg and Ljung[65], may result in considerably fewer parameters to obtain the same level of approximation as a model expanded in the backward shift operator z^{-1}. However, the efficacy of this approach depends on prior information regarding the accuracy of some dominant pole locations. The basis for the Laguerre expansions is the fact that for any stable transfer function $T(z)$, and any $a \in (-1, 1)$, there is a unique bounded real sequence α, such that

$$T(z) = \sum_{k=1}^{\infty} \alpha_k L_k(z, a)$$

where

$$L_k(z, a) = \frac{\sqrt{1 - a^2}}{z - a} \left(\frac{1 - az}{z - a} \right)^{k-1}$$

Observe that for $a = 0$, $L_k(z, 0) = z^{-k}$, which returns the usual expansion in the delay z^{-1}. The appropriate order of the expansion depends on the convergence properties of the partial sums. For

example, $\frac{1}{z-p}$ has a Laguerre expansion of order $n = 1$, provided that $a = p$. Since typically, p is not known, a good choice of n will depend on prior knowledge of p. For ARX models, replace H^{-1} and $H^{-1}G$ in the shift operator with Laguerre expansions. To pick a good Laguerre kernel requires either prior knowledge or else some data dependent means of selection. Another possibility is to select the kernel to reflect the desired closed-loop bandwidth.

An affine model set, *e.g.*, a Laguerre expansion for G, can also offset the issue of determining what is meant by a large data length. With this model, it is possible to precisely compute statistical properties without the need for either large model orders or large data lengths, *e.g.*, Kosut and Anderson[34]. However, another useful asymptotic property, also true for ARX models, is that if the input is white, then the first m impulse response coefficients of G are asymptotically unbiased, where m is the order of B_θ. Other useful results follow from this fact, *e.g.*, Aling and Kosut [1].

2.3 Ellipsoid Set-Membership Identification

2.3.1 Uncertain Non-parametric Dynamics

When an upper bound on the nonparametric model errors is known from prior knowledge, it is possible to compute a *parameter set* which is consistent with the data. Depending on the data, the parameter set is either an ellipsoid or an hyperboloid. In the latter case the data is considered to be "bad", that is, the spectral content of the data is concentrated too heavily at those frequencies where the nonparametric dynamics dominate. Thus, an ellipsoid indicates "good" data and there are several schemes for minimizing the size of these ellipsoids. Computation of the bounding ellipsoids is virtually no different than standard least-squares computations and can be accomplished in a batch or recursively. We plan to investigate efficient methods in our future work. Various kinds of prior information can also be included using the bounding ellipsoid approach. Some of these computational problems are generic, not specifically for robust control and identification, and are surveyed by Deller[14].

To see the main result more clearly, we can state the problem as follows: Use the measured input/output data

$$\{ y_t, u_t \mid t = 1, \ldots, N \} \tag{2.9}$$

to obtain a model set suitable for robust control design. To do this we need to make some assumptions. The first is that the system which produces the data is disturbance-free and linear time invariant. Thus,

$$y = Gu \tag{2.10}$$

where G has the (discrete-time) transfer function $G(z)$. Assume also that the true system is a member of the model set

$$\mathcal{G} = \{ G_\theta(1 + \Delta_G W_G) \ : \ \theta \in \Theta_{prior}, \|\Delta_G\|_{\mathcal{H}_\infty} \leq 1 \} \tag{2.11}$$

Thus, the model set consists of parametrized models with a multiplicative nonparametric error bounded by $W_G(z)$. The set Θ_{prior} represents the prior information by which the parameter vector is confined. We further characterize the parametric transfer function by using the standard ARX

form in Ljung[49]:

$$\begin{aligned} G_\theta(z) &= B_\theta(z)/A_\theta(z) \\ B_\theta(z) &= b_1 z^{-1} + \ldots + b_n z^{-n} \\ A_\theta(z) &= 1 + a_1 z^{-1} + \ldots + a_n z^{-n} \\ \theta &= (\ a_1\ \ldots\ a_n\ b_1\ \ldots\ b_n\ )^T \end{aligned} \tag{2.12}$$

The result in Kosut *et al.*[39] which forms the basis for the parameter set-membership estimation, is the following

Theorem 2.3.1 *Under the assumptions stated above, all parameters which are consistent with the measured data and the prior information are in the set*

$$\Theta_{prior} \cap \Theta_{wc}$$

where the "worst case equation error set" Θ_{wc} is defined by

$$\Theta_{wc} = \{\ \theta \in \mathbb{R}^p\ :\ \|A_\theta y - B_\theta u\|_N \leq \|B_\theta W_G u\|_N\ \}$$

($\|x\|_N = (\sum_{t=1}^N x_t^T x_t)^{1/2}$ is the usual ℓ_2-norm on $t \in [1, N]$.) The motivation for the term "worst case" refers to the fact that the nonparametric uncertainty Δ_G will take on the worst possible value such that $\|\Delta_G\|_{\mathcal{H}_\infty} \leq 1$. The set can be easily computed using least-squares methods and may be a hyperboloid, ellipsoid or the empty set depending on the data (see Kosut *et al.*[39]). Thus, the true system is guaranteed to be in the set:

$$\mathcal{G} = \{\ G_\theta(1 + \Delta_G W_G)\ :\ \theta \in \Theta_{prior} \cap \Theta_{wc}, \|\Delta_G\|_{\mathcal{H}_\infty} \leq 1\ \} \tag{2.13}$$

Instead of multiplicative model errors, we have also considered additive model error sets, i.e.,

$$\mathcal{G} = \{\ G_\theta + \Delta_G W_G\ :\ \theta \in \Theta_{prior}, \|\Delta_G\|_{\mathcal{H}_\infty} \leq 1\ \} \tag{2.14}$$

The resultant parameter set is then given by

$$\Theta_{wc} = \{\ \theta \in \mathbb{R}^p\ :\ \|A_\theta y - B_\theta u\|_N \leq \|A_\theta W_G u\|_N\ \}$$

Several other model error formulations can be used, e.g., inverse multiplicative, feedback and coprime factored. We will not discuss them here, but merely state them to indicate that this is a versatile approach which allows various kinds of prior information. More specific details and results using the set-membership approach are described in Kosut *et al.*[36], a copy of which is contained in this report as an Appendix.

2.3.2 Robust Control Design of Ellipsoid Sets

As a first step in using the ellipsoidal parameter set information, we simplified the robust control design problem to the case of FIR plants in an ellipsoidal set. Details can be found in Lau *et al.*[45, 46] which describe the continuous-time and discrete-time cases, respectively. Copies of these papers are contained in the Appendix.

We started with the simplifying assumption that the plant state-space description depended on uncertain parameters in the output matrix which are only known to lie in an ellipsoidal set. The desired control is chosen to minimize the maximum linear quadratic regulator (LQR) cost from all plants with parameters in the given set. Although no *a prior* form is assumed for the minimax control, it turns out that it is the LQR control for one of the plants in the set, *the worst-case plant*. By defining an appropriate operator mapping an element from the given ellipsoidal set to an element of the same set, the existence of this worst-case plant is proved. A simple algorithm is used to compute the worst-case plant.

The assumption that the output matrix in the plant description contains all the uncertainty deserves further discussion. First, this is a natural extension of the discrete FIR finite-horizon problem solved in Lau *et al.*[46]. In the continuous case, Laguerre models can be used so that the identification is reduced to estimating the Laguerre coefficients (see Wahlberg[64]). Uncertainty in the Laguerre coefficients can then be described by set membership of the output matrix. Second, by limiting uncertain parameters to the output matrix, we simplify the analysis and can gain more insights than if we had included parameter uncertainty in the plant dynamics also.

Specifically, we consider the following family of systems

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0 \tag{2.15}$$

$$y(t) = c^T x(t), \tag{2.16}$$

where A, b, and x_0 are fixed and given, and

$$c \in \Theta = \left\{\theta : (\theta - \theta_c)^T R(\theta - \theta_c) \leq 1, \; R = R^T > 0\right\}. \tag{2.17}$$

For a given control $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ and a fixed $c \in \Theta$, the LQR cost is defined to be

$$J(u, c) = \int_0^\infty [ru(t)^2 + y(t)^2] \, dt. \tag{2.18}$$

We assume that (A, b) is controllable (or at least stabilizable) and (c, A) is observable (or at least detectable) for all c in Θ. The robust control design problem is to find a control u that solves the following minimax problem:

$$\min_u \max_{c \in \Theta} J(u, c). \tag{2.19}$$

Since no *a prior* form is assumed for the control u, such as linear state-feedback, the minimization in (2.19) is over all possible $u : \mathbb{R}_+ \rightarrow \mathbb{R}$. Note also that we chose the initial time $t = 0$ for convenience only, the problem can be posed at any initial time $t = t_0$. Therefore, one can design a new controller each time Θ gets updated.

The cost objective in (2.18) and the ellipsoidal set in (2.17) lead to another interesting interpretation for the minimax problem in (2.19) once we rewrite (2.18) as

$$J(u, c) = \int_0^\infty [ru(t)^2 + x^T(t) c c^T x(t)] \, dt. \tag{2.20}$$

Now, instead of saying that we are designing a controller for a set of plants described by (2.15) through (2.17), we can also say that we are designing a controller for a set of objective functions. This interpretation contrasts with the standard LQR design where one controller is obtained for the selected weighting matrices. Therefore, the minimax control from (2.19) is less sensitive to how the states are penalized in the cost. This kind of control design method should be applicable to many practical situations as we seldom know exactly how much one state should be weighted against another.

2.3.3 Comparison with Stochastic Embedding

In the work of Goodwin *et al.*[24, 25, 26] a stochastic embedding philosophy is adopted which makes no assumptions on model order or data length. It is assumed that both the unmodelled dynamics and noise are drawn from a probabilistic set having certain amplitude and smoothness properties. These properties are then estimated by maximum likelihood techniques resulting in a set estimator.

Since all the trouble is related to that part of the system which is not modeled, *i.e.*, the "bias," it makes no sense to try to estimate the bias in the form of a parametrized model. That is tantamount to an additive high order plant model component which should have been incorporated in the plant model in the first place, *e.g.*, high-order ARX model sets.

To see the main idea, assume that the true system is described by (2.1), and an estimate $G_{\widehat{\theta}}$ of G has been obtained from

$$\widehat{\theta} = \arg\min_{\theta} \frac{1}{N} \sum_{t=1}^{N} (y - G_\theta u)_t^2$$

Since the model structure is incompatible with the true system, $G_{\widehat{\theta}}$ will be a biased estimate of G. We now make the assumption that the true system is the sum of a model in the model set and a *bias term which has an expectation value of zero:*

$$G(z) = G_{\theta_0}(z) + \Delta(z) \text{ with } \mathcal{E}\{\Delta(z)\} = 0$$

Here, the expectation is not taken over the data probability space, but over the unknown bias model set. In other words, the complicated problem of relating the bias to the data and the mismatch in structure of the true system and the model is avoided by simply assuming that the bias model is a zero-mean random variable. By modeling the bias in this rudimentary form, a bias model set parametrization is obtained which is described by a small number of parameters, yet is capable of representing a large set of error models.

As an example, assume that the expectation of the squared bias model impulse response is exponentially decaying:

$$\Delta(z) = \sum_{t=1}^{\infty} \beta_t z^{-t} \quad , \quad \mathcal{E}\{\eta_t^2\} = \alpha\rho^t$$

where $0 < \rho < 1$. Thus, the bias model set is described by only two parameters α and ρ. With some additional assumptions, *e.g.*, gaussianity and G_θ an affine Laguerre expansion, an explicit formula of the Fisher information matrix can be derived which forms the basis for an optimization procedure. Hence, the two parameters which describe the general shape and size of the less certain part of the system model can be directly estimated from the data.

2.4 $\mathcal{L}_\infty$ Identification

In this section, a new criterion for system identification is introduced, which we loosely call $\mathcal{L}_\infty$-identification. At the present time, very little is known about this approach, and hence, we can only guess about the consequences for set-membership identification and the corresponding robust controller design. However, like LS, this approach also leads to solving a convex optimization problem. Unlike LS, it does not appear at this time that the solution can be expressed in closed-form. However, the criterion is a convex function, so therefore, numerical methods will reliably compute the solution, specifically, interior point methods. In the future we hope to further understand

the properties of this estimator and develop reliable computational methods. Hopefully, this new methodology will result in more natural set estimators suitable for robust control design.

The parametric approach to system identification is based on selecting an appropriate model structure and a search for the parameters of the model that best describes the data. Usually, the best model within the model set is characterized as the one that minimizes a selected norm of the prediction errors, usually the 2-norm. In this section a new norm is introduced. Minimizing this norm is equivalent, asymptotically, to minimizing the supremum of the spectrum of the prediction error over all frequencies, or equivalently minimizing its $\mathcal{L}_\infty$ norm.

Given a scalar finite sequence $\{e_i,\ i = 1, \ldots, N\}$ which represents the prediction errors computed from the observed data and a guessed model parameter vector θ. Based on this sequence, form the $(N + M - 1) \times M$ matrix,

$$E_{NM} = \frac{1}{\sqrt{N}} \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ e_2 & e_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_M & e_{M-1} & \ddots & e_1 \\ \vdots & \vdots & \ddots & \vdots \\ e_N & e_{N-1} & \cdots & e_{N-M+1} \\ 0 & e_N & \cdots & e_{N-M+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_N \end{pmatrix} \tag{2.21}$$

with $1 \leq M \leq N$. Note that E_{NM} is constant along the diagonals, and for $M = 1$, E_{N1} is a column vector with $e_i/\sqrt{N}$ as its elements. Denote this vector by E_N. Hence, the matrix E_{NM} is completely specified when $E_N (= E_{N1})$ is given.

Define the new norm as the maximum eigenvalue of $E_{NM}^T E_{NM}$,

$$V_M(E_N) = \overline{\lambda}\left(E_{NM}^T E_{NM}\right) = \overline{\sigma}^2\left(E_{NM}\right) \tag{2.22}$$

where $\overline{\lambda}(F)$ denotes the maximum eigenvalue of F and $\overline{\sigma}(F)$ denotes the maximum singular value of F. Note that for $M = 1$, $V_M(E_N)$ is the usual quadratic norm. ¿From Grenander and Szego[28], we obtain the following limiting properties:

$$\lim_{N\to\infty} E_N^T E_N = \frac{1}{2\pi}\int_{\pi}^{\pi} S_{ee}(\omega)d\omega \tag{2.23}$$

$$\lim_{M\to\infty}\left(\lim_{N\to\infty} \overline{\sigma}^2(E_{NM})\right) = \sup_{|\omega|\leq\pi} S_{ee}(\omega) \tag{2.24}$$

$$\lim_{M\to\infty}\left(\lim_{N\to\infty} \underline{\sigma}^2(E_{NM})\right) = \inf_{|\omega|\leq\pi} S_{ee}(\omega) \tag{2.25}$$

where we assume that N goes to infinity faster than M.

Relation (2.24) is very illuminating and shows that by minimizing V_M as M approaches infinity, the supremum of the spectrum of the prediction error over all frequencies is minimized. Because of this property, we referred to the identification problem using the new norm as the $\mathcal{L}_\infty$ *identification problem*. In contrast, by minimizing the usual quadratic norm, the integral of the spectrum of prediction error over all frequencies is minimized (see Ljung[49]), and this can be referred to as

$\mathcal{L}_2$ identification problem. It seems plausible, that this norm is potentially very useful for robust control design. More details can be found in Massoumnia and Kosut[51] which is included in the Appendix.

Chapter 3

The Future: A Graphical User Interface for System Identification

A long range objective of the present work is the development of some mathematical and computational tools that are appropriate to the next generation of CACSD (Computer Aided Control System Design) environments. These future CACSD packages will be radically different from the present packages in that they will truly be able to perform control systems synthesis and rapid prototyping, rather than just analysis and simulation.

In our view of the future, the engineer will commence the design with uncertain and/or incomplete information consisting partly of prior knowledge, measured data, and a set of closed-loop design objectives and constraints. Once this information is fed into the CACSD program, it will in turn generate controllers that meet the performance requirements while respecting the constraints, or else inform the engineer that the constraints cannot all be satisfied, suggest some trade-offs as well as alternative experiments to obtain data which may reduce uncertainty. As the engineer thinks of more constraints and requirements, and/or obtains more data, these are entered into the computer and are accounted for as they are entered. Thus, the CACSD process is still interactive, but the level of interaction with the computer is much higher than it is at present. Moreover, the interactive use of real data would be much more possible than at present.

In order for this ideal situation to come about, it is necessary first to solve some important mathematical and computational problems residing in the interface between controller implementation on the actual system and controller design based on a model of the system.

System identification is a typical example of an iterative inter-active procedure where several results have to be computed, analyzed and re-iterated again with modified design parameters. In order to do this, the user repeatedly has to enter a sequence of commands for computing frequency responses, spectral density functions and prediction error norms. Even in high-level interactive CACSD programs like MATRIX$_X$ and MATLAB it is virtually impossible to execute this procedure without having to write command files for each specific task. Figure 3.1 shows typical paths and functions in the MATRIXx system identification environment. Instead of concentrating on the design task, the user is mainly occupied with designing, organizing and maintaining a large number of specific programs for standard procedures. As a conclusion, the current CACSD software is inadequate for most users, both in the sense of user-friendliness and software design capabilities.

As an example, at Integrated Systems Inc. (ISI), we have recently introduced the XMATH

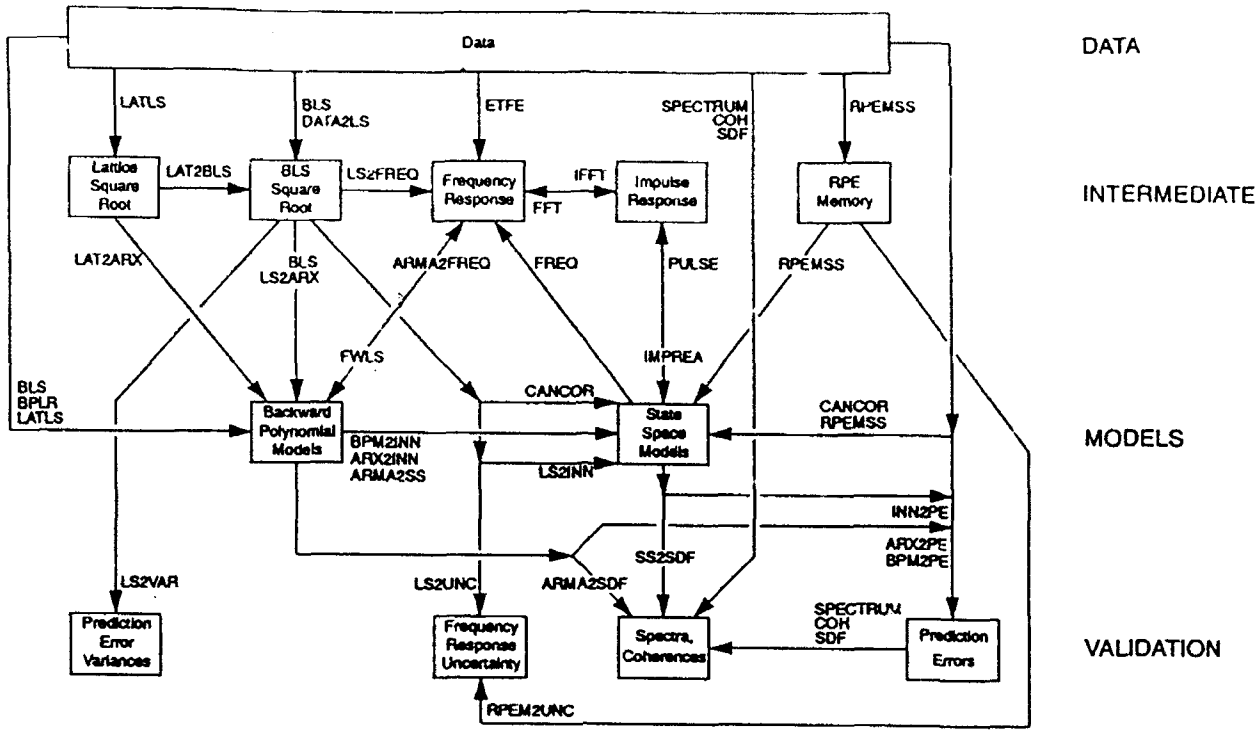

Figure 3.1: MATRIXx System Identification command overview

product, which provides an ideal platform for the development of portable window-based CACSD software such as system identification and control design. The important difference with the current interactive CACSD programs in terms of user interface is that XMATH incorporates an interactive X-windows based GUI development tool. This makes it possible to efficiently design interactive mouse-driven application software where the interaction takes place through one or more specially suited windows for each of these tasks. Such windows display all relevant parameters, as well as graphical output like frequency response plots and bar graphs of error norms as a function of model order. Standard validation and identification options are activated by a pulldown menu with on-line help, and all displayed parameters are open to be changed for quick recomputation of the results.

As an example, consider the window displayed in Figure 3.2 which was written in XMATH/GUI and which is intended for interactive system identification. This tool allows the user to identify all ARX models up to a certain order, view their frequency response and confidence intervals, and vary the data window (gray area in the data plot area) and model order (gray bar in the two upper right error norm plots) using the mouse only. In the lower left area, all important model parameters are displayed and various options can be accessed by activating a pulldown menu from the top menu bar.

Clearly the XMATH-GUI can be used for the development of an interactive object oriented environment which is sensible for a wide variety of users in the field of system identification and control system design. This not only relieves the user of the burden of command syntaxes, but also makes the design procedure completely self-explanatory.

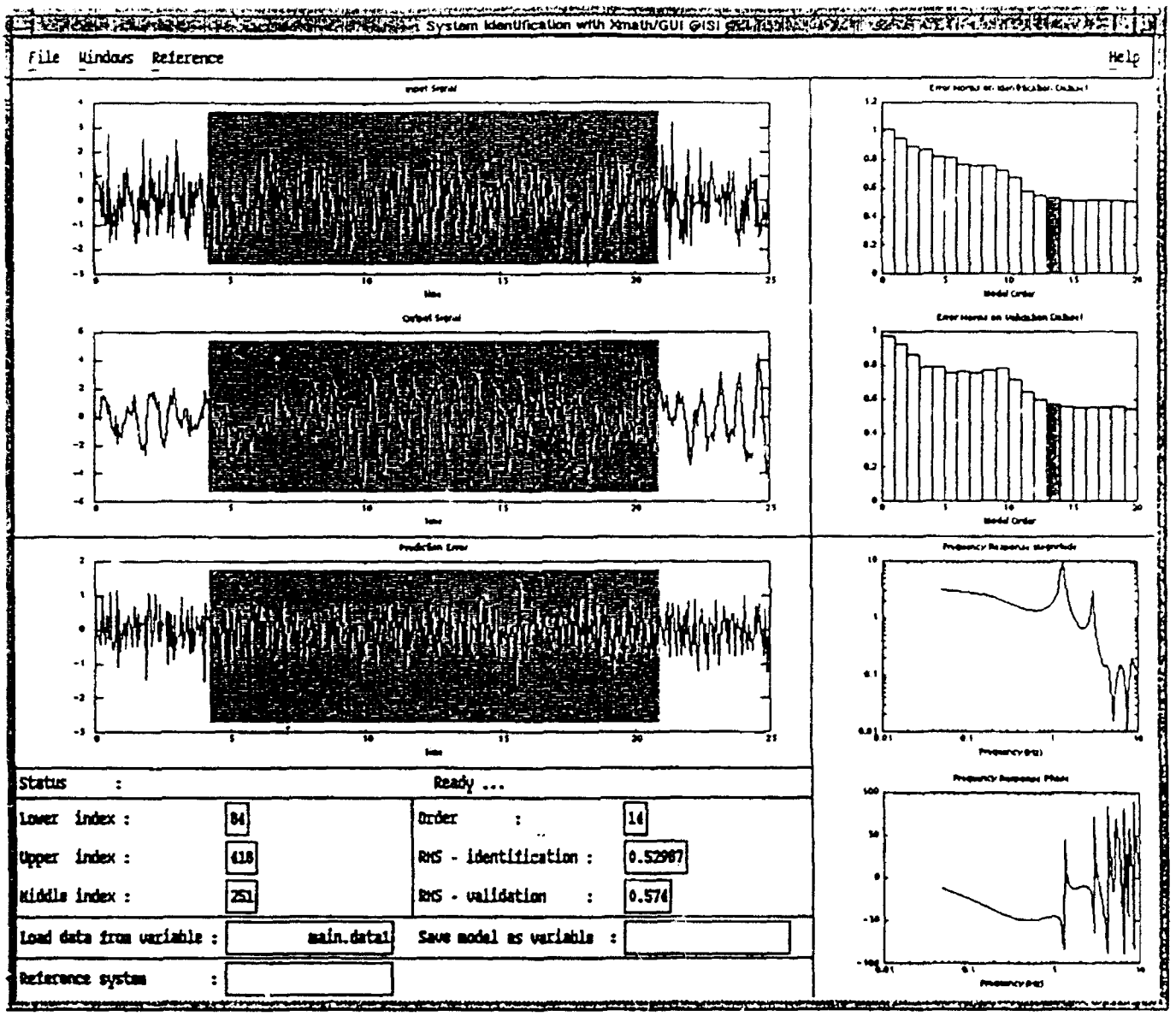

Figure 3.2: An example XMATH/GUI window

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Appendix A

Preprints of Papers

A selection of papers which have or will appear in both journals and conference proceedings is included in this appendix.

On the Character of Uncertainty for System Identification and Robust Control Design *

Robert L. Kosut†

September 15, 1992

"It ain't the things you don't know what gets you in deep trouble. It's the things you know for sure, but what ain't so." –Uncle Remus.

Nothing could more aptly describe the predicament when faced with the problem of designing a controller from accumulated sensed input-output data. The identification, or estimation, of a system's transfer function from input-output data has a long history and there are many excellent survey articles and textbooks that can be referenced, *e.g.*, [4], [8], [7],[15], [14]. The problem with all the methods discussed in these references, insofar as robust control design is concerned, is that model error estimates are usually not available, and if available, cannot be trusted. The principal reason for this difficulty is that the identification algorithms are developed under the false assumption that the true system is in the model set. As a result, the model estimate, loosely speaking, is "biased", and hence, a controller designed using the estimate may result in unacceptable closed-loop behavior, a phenomenon which is well documented, *e.g.*, [14, 3, 1]. To paraphrase the above aphorism, "Trouble is bound to follow if the identified model is *known for sure* to be the true system."

This situation is unfortunate, because all the standing assumptions made in current robust control design methods require a model set description which typically consists of a nominal model and an error estimate, usually a norm bound, where both together are *guaranteed* to encompass the true system. To fulfill the needs of robust control design will therefore require a new approach to system identification which provides both a nominal model and a measure of its uncertainty. Such schemes have been referred to by various names, *e.g.*, set-membership identification, set-estimation, uncertainty modeling, as well as other self-canceling phrases -- how does one model an uncertainty? This research topic has received strong interest recently as evidenced by this workshop, the recent special issue [10], and the many conference sessions planned at the next ACC and CDC.

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† Integrated Systems, Inc., 3260 Jay St., Santa Clara, CA, 95054 and Department of Electrical Engineering, Stanford University, Stanford, CA.

Formulating the Problem

"If what is said is not meant, then what ought to be done, remains undone."
– Confucious.

Sometimes solving a problem means finding a simple or direct statement of the problem in the first place. In attempting to distill the problem formulation to its essence, perhaps it is this: given a finite collection of sensed sampled input/output data from an unknown system, what level of confidence can be assigned to a feedback controller design or modification? If, other than the measured data, there is no additional knowledge about the system, then the problem is solved: there is no safe controller. Anything can happen, because there is no means for inferring the future from the past. Therefore, to make the problem meaningful, it is necessary to make *a priori* assumptions about the system. These assumptions can be either qualitative or quantitative. For example, assuming that the unknown system is linear-time-invariant is *qualitative a priori* knowledge. Knowing that it is stable can still be classified as qualitative, but assigning a region for pole locations or knowing a bound on the impulse response is *quantitative*. A similar classification can be made regarding signal charateristics. Knowing that a signal is white is qualitative; but knowing a precise value for the variance is quantitative.

Although *a priori* quantitative information may be readily available, *e.g.*, from the underlying physics, I think that it is first necessary to resolve the more pristine problem of specifying a minimal amount of qualitative *a priori* data so as to assign a high degree of confidence to a controller design.

Is Nature Good, Evil, or Indifferent?

The phrase "high degree of confidence" needs clarification. Do we mean worst-case or high probability?

Current robust control formats are based on worst-case scenarios. Nature is perceived as Evil, and hence, does the *wrong* thing, from our perspective. However, if this is not the case, and Nature is at worst Indifferent or Neutral, then the problem should be posed in reverse: to fulfill the needs of system identification, long resting on a probabilistic (neutral) foundation, may require a new approach to robust control which allows for a probabilistic description of uncertainty! This latter possibility invokes the current debate on the intrinsic nature or character of the uncertainty set. Is it probabilistic or worst-case deterministic? Clearly both can be used to quantify uncertainty in either disturbances and transfer functions. However, searching for the worst-case may be a hopeless task. If the worst-case has not yet occurred, it might in the future, and hence, the search never ends. Fitting a probablistic model is more sensible in this regard, but a 99.99% confidence level does not preclude the remaing .01% from occurring.

A probabilistic, or stochastic, description of a disturbance is common practice and forms the basis for $\mathcal{H}_2$-filtering and control design, *i.e.*, optimal filtering and LQG control design

[2]. A power bounded set of disturbances and/or a worst-case deterministic description of transfer function uncertainty leads to $\mathcal{H}_\infty$ methods of control design, *e.g.*, [5]. These sets can be combined leading to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design, *e.g.*, [9]. The above examples by no means exhaust the possible deterministic and probabilistic sets. For example, sequences can be uncertain but have a bounded spectrum or a bounded magnitude. Transfer functions can be uncertain but with (time) bounded impulse responses, and so on. The choice of which uncertainty characterization to use depends upon prior knowledge about the true system. Clearly different assumptions ought to lead to set estimators with differing forms and mixtures of probabilistic and/or worst-case deterministic uncertainty types.

As a case in point, if we begin with a stochastic description of the exogenous inputs to a system, then the standard least-squares based identification method with a high-order ARX model structure leads naturally to a *purely parametric* uncertainty which, depending on further assumptions, is either probabilistic (normally distributed) or worst-case deterministic (ellipsoid bounded), *e.g.*, [12, 11, 6]. To conform to current robust control paradigms, the parametric characterization of uncertainty must be transformed to a non-parametric worst-case deterministic frequency domain bound, a transformation that is not without a considerable loss of information. Dealing directly with the worst-case deterministic (ellipsoid bounded) parameter uncertainty leads to some new insights into robust control design *e.g.*, [13]. For the probabilistic form of parameter uncertainty, it is my view that it would be better to develop a compatible theory of "probabilistic" robust control.

Going in this direction, however, immediately raises the question: what does a robust control mean in the context of probabilities? We tend to think of a robust controller as providing an absolute guaranty against instability and/or certain levels of performance degradation given a deterministic, or "hard " bound on plant uncertainty. With a probabilistic description, or "soft" bound, we must decide if 99.99% is safe enough. To turn the question the other way, the deterministic bounds necessitates guarding against the worst-case, which may be extreme, *i.e.*, unlikely, thereby leading to an conservative controller. But this brings us back to exactly the question of probabilities and outcomes, and finally to a more fundamental question: is Nature neutral or conspiratorial?

Towards a New Paradigm, or Paradigm Lost

Attempting an answer may not be necessary, nor very fruitful. I think that a better attitude at this point is to follow the consequences, without prejudice, of developing a theory of set-membership identification and corresponding "robust" control design methods compatible with probabilistic plant set descriptions. This to me seems the more sensible engineering oriented character of uncertainty.

Hopefully, as a result of research efforts in many different directions, new paradigms will arise which combine system identification and robust control design. With the wide availability and use of CACSD packages, the benefits of this research could be widely utilized in many engineering fields. Hence, it becomes imperative that the resulting methodologies are comprehensible and useful for the engineering community at large; not just understandable

to a few experts. The onus is on us!

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Set-Membership Identification of Systems with Parametric and Nonparametric Uncertainty

Robert L. Kosut, *Fellow, IEEE*, Ming K. Lau, *Student Member, IEEE*, and Stephen P. Boyd, *Member, IEEE*

Abstract—**A method is presented for parameter set estimation where the system model is assumed to contain both parametric and nonparametric uncertainty. In the disturbance-free case, the parameter set estimate is guaranteed to contain the parameter set of the true plant. In the presence of stochastic disturbances, the parameter set estimate obtained from finite data records is shown to have the property that it contains the true-plant parameter set with probability one as the data length tends to infinity.**

I. Introduction

In the traditional adaptive control system, the identified model is used for on-line controller design without any regard for errors between this model and the true system which generated the data. The identified model is usually selected out of a model set with unknown parameters as depicted in Fig. 1. The controller is designed as if the parameter estimates were in fact the correct parameters for describing the plant. This is known as applying the certainty equivalence principle. In the ideal case, it is assumed that there exist parameters, which if known, would precisely account for the measured data. Even in this ideal case, the transient errors between the identified model and the true system can be so large as to completely disrupt the performance. In the usual (nonideal) case, the true system is not in the model set, therefore, both unacceptable transient or asymptotic behavior can occur, e.g., [1].

Following the ancient Greek adage,[1] "*Well begun, half done*," one ought to construct, at the outset, an adaptive control system which specifically accounts for the inevitable model error, i.e., an adaptive *robust* control. Depicted in Fig. 2 is our proposed scheme where the traditional parameter estimator is replaced with an estimator that produces a model set. Thus, point estimation of a single model is replaced with *set-membership identification*. The estimated model set can contain both parametric and nonparametric descriptions of uncertainty arrived at from both measured and prior data.

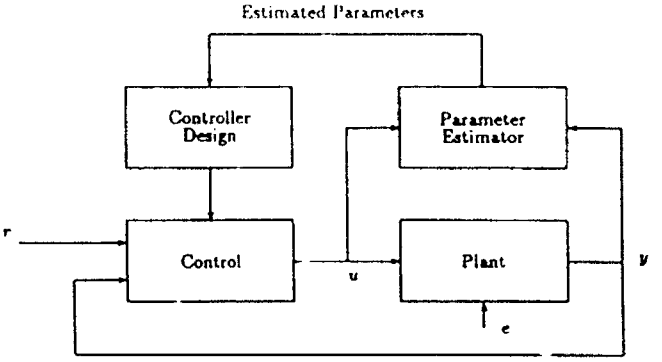

Fig. 1. Traditional adaptive control system with parameter estimator.

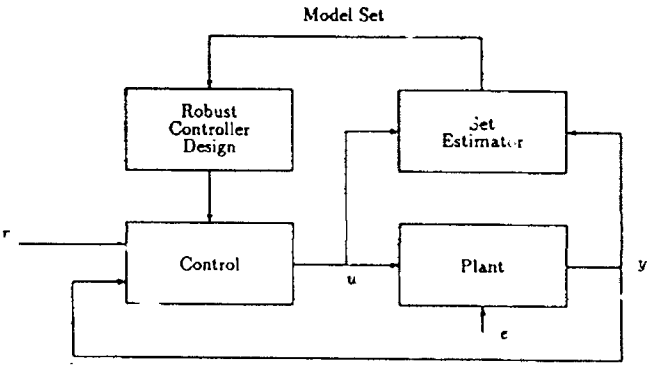

Fig. 2. Adaptive control with set estimator.

We also replace the traditional controller design algorithm with a robust controller design algorithm which accepts the model set format. By referring to a robust controller we mean a controller that achieves some specific set of specifications for any plant model in the model set. The robust controller design thus takes a set of models as input and produces a controller that is guaranteed to meet the specifications for all models in this set. The robust controller design can also report the worst-case performance with respect to the model set. It is also true that if the model set is too large, or the specifications are too tight, then no robust controller will exist.

During the transient or learning phase, the estimated model set could be a poor representation of the true system as it could be quite large. However, if the system which generated the measured data is contained in the estimated set, the robust controller will be stabilizing, though may be of low author-

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R. L. Kosut is with Integrated Systems, Inc., Santa Clara, CA 95054 and with the Information Systems Laboratory, Stanford University, Stanford, CA 94305.

M. K. Lau and S. P. Boyd are with the Information Systems Laboratory, Stanford University, Stanford, CA 94305.

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[1] Αρξη ημισυ παντων. Literally translated (The) beginning (is the) half of all [16].

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ity. Conversely, if the model set becomes smaller after some time, this will be reflected in a higher authority controller with more desirable performance characteristics.

It is important to point out, and even emphasize, that although this approach is inspired by a separation principle, *it is not optimal*. Roughly speaking, set estimation and robust controller design might benefit from being coupled. For example, the input u might be temporarily manipulated in such a way so that the set estimator could rapidly learn and therefore improve future performance at the expense of current performance. In a purely Bayesian framework, notions of optimality along this line are made precise in [9].

Although not guaranteed to be optimal, the scheme shown in Fig. 2 is at least less heuristic than the traditional scheme of Fig. 1. For example, if the set estimator is *consistent*, that is, the true plant is in the estimated model set, and moreover, if we stop adapting at any given point, then we are guaranteed a worst-case performance as reported by the robust controller design.

In this paper, we address the problem of parameter set estimation where the system model contains both parametric and nonparametric uncertainty. In our formulation, we use the measured data to delineate a parametric set which accounts for *a priori* knowledge of nonparametric dynamics and disturbances. Observe that if measured data is not used, then the identified model set consists of a constant model set and the "adaptive" controller reduces to a single robust design. We can also recover the traditional adaptive scheme by replacing the robust design with a heuristic design which uses a typical model in the set, e.g., the "center" or "average" model.

We will not address the robust control design issues as different methodologies for robust control design, particularly for plants with uncertain nonparametric linear dynamics, can be found in [26], [8], and [12]. Methods for robust control design of plants with parametric uncertainty are described in [2], [5] and the references therein. In the case of parametric set-membership uncertainty, minimax controllers are considered in [22] and [21].

At present, there are several competing and complementary methodologies for the design of set estimators, e.g., [29], [20], [17], [14], [18], and [32]. Related work on the limitations of identification of linear-time invariant systems can be found in [13], [15], [24], and [28]. Our work here follows closely to that described in [31], [32], and [18] for the disturbance-free case with nonparametric uncertainty, and in [23] for the disturbance case. The parameter sets developed here are similar in form to those developed in [10], [11], [25], and [3] for the case with no nonparametric uncertainty but with bounded disturbances. The foundation and impetus for much of the work in parameter set-membership identification can be traced back to [27], and [4] for the state-estimation problem.

The paper is organized as follows. After introducing some notation and standard definitions in the next section, the problem is formulated in Section III. Parameter set estimates for the disturbance-free equation-error case are developed in Section IV. In the presence of stochastic disturbances, equation-error parameter set estimates computable from finite data records are presented in Section V. Extensions to the output-error case and deterministic disturbances are discussed in Section VI. The paper concludes with some remarks in Section VII.

II. Notation and Preliminaries

Transfer Functions: In this paper, we consider sampled-data systems with transfer functions in the complex variable z. If the system is denoted by G, then its transfer function is denoted by $G(z)$. Typically, $G(z)$ is obtained as the zero-order hold equivalent of a continuous-time transfer function $P(s)$. Thus,

$$G(z) = \mathcal{Z}\mathcal{O}\mathcal{H}\{P(s)\} \tag{1}$$

$$\triangleq (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s}P(s)\right\} \tag{2}$$

where $\mathcal{Z}\mathcal{O}\mathcal{H}\{\cdot\}$ and $\mathcal{Z}\{\cdot\}$ denote the zero-order hold and the usual z-transform operations, respectively.

A transfer function $G(z)$ is *stable* if all its poles are strictly inside the unit circle $|z| = 1$. The frequency response of $G(z)$ is the function $G(e^{j\omega})$ restricted to the domain $|\omega| \le \pi$, where ω is the frequency variable normalized with respect to the sampling frequency. For a stable transfer function $G(z)$, the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms are defined as

$$\|G\|_{\mathcal{H}_\infty} \triangleq \sup_{|\omega| \le \pi} |G(e^{j\omega})| \tag{3}$$

$$\|G\|_{\mathcal{H}_2} \triangleq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 \, d\omega\right)^{1/2}. \tag{4}$$

Sequences: A sequence x is a function of discrete-time points, i.e., $x: \mathbb{N} \to \mathbb{R}^p$ where $\mathbb{N} = \{1, 2, \cdots\}$ is the set of positive integers. We write $x(t)$ to mean the value of the sequence at a particular time t, normalized with respect to the sampling interval. Hence, time takes on integer values with initial time defined as $t = 1$.

Following [24], a sequence x is *quasi-stationary* if $\mathcal{E}(x(t))$ is bounded for all t and its autocorrelation

$$r_{xx}(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathcal{E}\left(x(t)x(t-\tau)\right) \tag{5}$$

exists for all integers τ, where $\mathcal{E}(\cdot)$ denotes the expectation operator. If x is a deterministic sequence, the expectation is without effect and quasi-stationary then means that x is a bounded sequence such that the limits

$$r_{xx}(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} x(t)x(t-\tau) \tag{6}$$

exist. For easy notation, we introduce the symbol $\bar{\mathcal{E}}$ by

$$\bar{\mathcal{E}}(x) \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathcal{E}\left(x(t)\right). \tag{7}$$

The power spectrum of x is defined as

$$S_{xx}(\omega) \triangleq \sum_{\tau=-\infty}^{\infty} r_{xx}(\tau)e^{-j\omega\tau} \tag{8}$$

This leads to the power in x given by

$$r_{xx}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)\,d\omega = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathcal{E}\left(x(t)^2\right). \tag{9}$$

Similar definitions apply to the cross spectrum $S_{xy}(\omega)$ of the sequences x and y.

The sample-mean operator $\mathcal{E}_k(\cdot)$ is defined to be

$$\mathcal{E}_k(x) \triangleq \frac{1}{k}\sum_{t=1}^{k} x(t). \tag{10}$$

We use $\|x\|_{k2}$ to denote the truncated l_2-norm of a sequence

$$\|x\|_{k2} \triangleq \left(\sum_{t=1}^{k} x(t)^2\right)^{\frac{1}{2}} \tag{11}$$

hence,

$$\mathcal{E}_k(x^2) = \frac{1}{k}\|x\|_{k2}^2. \tag{12}$$

Linear Operators: The notation Gx means the sequence obtained when the system G operates on the sequence x. We write $(Gx)(t)$ to mean the value at time t of the sequence Gx.

When we say that G is a linear-time invariant system, we mean that Gx is the convolution operation

$$(Gx)(t) = \sum_{k=0}^{t-1} g(k)x(t-k) \tag{13}$$

where the sequence g is the pulse response of G. Thus, G has the transfer function

$$G(z) = \sum_{k=0}^{\infty} g(k)z^{-k}. \tag{14}$$

The above definition restricts the sequence Gx to $t \geq 1$. Hence, the system G can be regarded as having no memory of events prior to $t = 1$, the initial time. Roughly, this means all initial conditions are zero.

To reduce notation, we use the transform variable z to denote the shift operator, so $z^k x(t) = x(t+k)$, $z^{-k}x(t) = x(t-k)$, and $z^k x$ shifts each member of the sequence x.

III. Problem Formulation

The problem is to use the measured sampled data

$$\{y, u: t = 1, \cdots, N\} \tag{15}$$

to identify a model set suitable for robust control design. The system which produced the data is assumed to be a linear-time invariant system of the form

$$y = Gu + v \tag{16}$$

where G is a linear-time invariant system with transfer function $G(z)$, u is an applied input, y is the measured output, and v is a disturbance as seen at the output. It is also assumed that both y and u have finite power, that is, $r_{yy}(0) < \infty$ and $r_{uu}(0) < \infty$.[2]

A. Model Set Assumptions

The model set $\mathcal{M}$ is defined as follows:

$$\mathcal{M} \triangleq \{y = Gu + v: G \in \mathcal{G}, v \in \mathcal{V}\} \tag{17}$$

where $\mathcal{G}$ is the set of linear-time invariant systems and $\mathcal{V}$ is the set of disturbances. It is assumed that the true system (16) is a member of the model set $\mathcal{M}$. The reader should be cautioned that G defined in the model set $\mathcal{M}$ is *not* the same as G in (16). To avoid adding more subscripts G_{true}, etc., unless otherwise stated as part of some set, e.g., $G \in \mathcal{G}$, the symbols G, y, u, and v refer to the true system (16).

We first concentrate on the disturbance-free case, i.e., $v = 0$, in the next section. The disturbance set $\mathcal{V}$ is discussed later in Section V.

The set of linear-time invariant systems is defined by

$$\mathcal{G} \triangleq \{G_\theta(1 + \Delta_G W_G): \theta \in \Theta_{\text{prior}}, \|\Delta_G\|_{\mathcal{H}_\infty} \leq 1\} \tag{18}$$

where $G_\theta(z)$ is a parametric transfer function with parameters $\theta \in \Theta_{\text{prior}}$, referred to as the *prior parameter set*. The system $\Delta_G W_G$ is referred to as the *multiplicative nonparametric uncertainty*. It is a dynamic uncertainty characterized by an uncertain but unity bounded stable-transfer function $\Delta_G(z)$ and a known stable-transfer function $W_G(z)$. Note that $W_G(z)$ acts as a frequency weighting function, whose frequency response magnitude $|W_G(e^{j\omega})|$ reflects the size of the nonparametric uncertainty. Since a parametric model of a system is never complete unless we have some idea on its limitations and accuracies, we assume that the uncertainty weighting function $W_G(z)$ is known. Having knowledge of W_G is precisely the assumption made in robust control design, e.g., [8]. However, the center of the model set is fixed in robust control design, here it is parametric, i.e., G_θ.

Suppose the true system G is in $\mathcal{G}$ and we are interested in all the possible representations of G in $\mathcal{G}$. Solving for Δ_G in (18) in terms of G and θ, we get

$$\Delta_G = \frac{G - G_\theta}{W_G G_\theta}. \tag{19}$$

We define

$$\Theta^* \triangleq \left\{\theta: \left\|\frac{G - G_\theta}{W_G G_\theta}\right\|_{\mathcal{H}_\infty} \leq 1\right\} \tag{20}$$

and refer to Θ^* as the *parametric limit set* because it does not depend on the data set but rather on the true but unknown system G. As a result, $\Theta^* \cap \Theta_{\text{prior}}$ is the set of all possible parameter values consistent with the assumption that the true system G is in $\mathcal{G}$. Consequently, it is not possible to consider a "true" parameter value because any member of

[2] Input and output sequences with finite power occur, for example, when G is stable and u has finite power, or when G, not necessarily stable is stabilized by an appropriate feedback and the exogeneous inputs to the feedback system have finite power.

$\Theta^* \cap \Theta_{\text{prior}}$ is a possibility since the decomposition of G into G_θ and Δ_G is not unique. Thus, the goal is to obtain an estimate of the set Θ^* from the measured data.

Throughout the remainder of the paper we further characterize the parametric transfer function $G_\theta(z)$ by using the standard ARX form [24]

$$\begin{aligned} G_\theta(z) &= B_\theta(z)/A_\theta(z) \\ B_\theta(z) &= b_1 z^{-1} + \cdots + b_m z^{-m} \\ A_\theta(z) &= 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \\ \theta &= [a_1 \cdots a_n \; b_1 \cdots b_m]^T. \end{aligned} \tag{21}$$

Thus, the parameters are the coefficients in the parametric transfer function. With this parametrization, the limit set becomes

$$\Theta^* = \left\{ \theta : \left\| \frac{A_\theta G - B_\theta}{W_G B_\theta} \right\|_{\mathcal{H}_\infty} \le 1 \right\}. \tag{22}$$

The problem we are addressing in this paper is to find an estimate of Θ^*. We should also point out that other than what is assumed for the transfer function $\Delta_G(z)$, we do not estimate it from the data. We first give an example of Θ^*, and then in the next section, describe a set estimator in the disturbance-free case.

B. Example of Limit Set

Suppose that the true transfer function is

$$G(z) = \mathcal{ZOH}\left\{ \left(\frac{10}{s+1} \right) \left(\frac{10^2}{s^2 + 2(0.005)(10)s + 10^2} \right) \right\}. \tag{23}$$

The sampling frequency is chosen to be $2\pi(10)$ rad/s or 10 Hz. Observe that the system has a simple pole at 1 rad/s, and a very lightly damped resonance at 10 rad/s. Suppose we are interested in obtaining a good low-frequency model by neglecting the resonance, but accounting for it as one realization of some nonparametric dynamics. Thus, select the parametric transfer function as

$$G_\theta(z) = \frac{bz^{-1}}{1 + az^{-1}}, \quad \theta = \begin{bmatrix} a \\ b \end{bmatrix}. \tag{24}$$

Consider the following weights:

$$W_{G,1}(z) = 65 \left| \mathcal{ZOH}\left\{ \frac{s+1}{s+5} \right\} \right|^4 \tag{25}$$

$$W_{G,2}(z) = W_{G,1}(z) - 65\left(\frac{1}{5}\right)^4. \tag{26}$$

Either of these weights can account for the resonance, but they reflect different prior low-frequency uncertainties. The weight $W_{G,1}$ reflects a low-frequency multiplicative uncertainty of about 10% where it has a dc gain of about 0.1, and it anticipates a rather large resonance at frequencies beyond about 10 rad/s where the magnitude of $W_{G,1}$ is greater than 100. $W_{G,2}$ is essentially the same but has a zero dc gain. Shown in Fig. 3 are the frequency response magnitudes and the multiplicative error with respect to a "nominal" parametric transfer function

$$G_{\theta_{\text{nom}}}(z) = \mathcal{ZOH}\left\{ \frac{10}{s+1} \right\}. \tag{27}$$

With the sampling frequency of 10 Hz,

$$\theta_{\text{nom}} = \begin{bmatrix} a_{\text{nom}} \\ b_{\text{nom}} \end{bmatrix} = \begin{bmatrix} -0.9048 \\ 0.9516 \end{bmatrix}. \tag{28}$$

This transfer function can be viewed as an approximation of $G(z)$ obtained by neglecting the resonance in (23). Remember, there is no true parameter value, rather, there is a true set Θ^*, one element of which is this nominal parameter value.

Points in the limit set corresponding to the above weights are shown in Fig. 4. These points are obtained by testing θ in (19) over a set of points. If a point's corresponding Δ_G satisfies $\|\Delta_G\|_{\mathcal{H}_\infty} \le 1$, then it belongs to Θ^*. Since $W_{G,2}(e^{j\omega})$ is zero at $\omega = 0$, i.e., the dc gain of $G(z)$ is assumed known, and the two parameters in θ are constrained to lie on a line in the parameter space. The line becomes "blurred" in the limit set corresponding to $W_{G,1}$ because there is no frequency where the frequency response of $W_{G,1}$ is identically zero.

IV. Disturbance-Free Equation-Error Set Estimation

In the disturbance-free case, we have $v = 0$. Thus, the model set in (17) reduces to

$$\mathcal{M} = \{ y = Gu : G \in \mathcal{G} \} \tag{29}$$

with $\mathcal{G}$ given by (18).

Theorem 1: Suppose the measured data $\{y, u : t = 1, \cdots, N\}$ is generated from $y = Gu$ with $G \in \mathcal{G}$. Then the following holds:

$$\Theta^* \subseteq \Theta[N] \subseteq \Theta_k, \qquad \forall k \in [1, N], \forall N \in \mathbb{N} \tag{30}$$

where $\Theta[N]$ and Θ_k are given by

$$\Theta_k \triangleq \{ \theta : \| A_\theta y - B_\theta u \|_{k2} \le \| W_G B_\theta u \|_{k2} \} \tag{31}$$

$$\Theta[N] \triangleq \bigcap_{k=1}^{N} \Theta_k. \tag{32}$$

Remarks: We refer to Θ_k or $\Theta[N]$ as *equation-error parameter sets* because the equation-error term $A_\theta y - B_\theta u$ appears in the definition [24]. Observe that the equation-error sets depend only on the measured data and the known bounding transfer function $W_G(z)$. Because Θ^* is a subset, it follows that Θ_k for any $k \in [1, N]$ or $\Theta[N]$ is an estimate of Θ^*. These sets are easy to compute as will be shown in Section IV-C. First we prove the theorem.

Proof: First, recall the following fact from [7]. If T is a stable linear-time invariant operator with transfer function

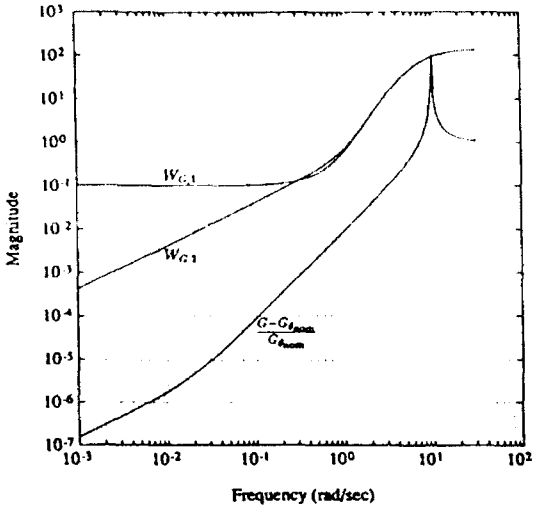

Fig. 3. Frequency response magnitudes of $W_{G,1}$, $W_{G,2}$, and $(G - G_{\theta_{nom}})/G_{\theta_{nom}}$.

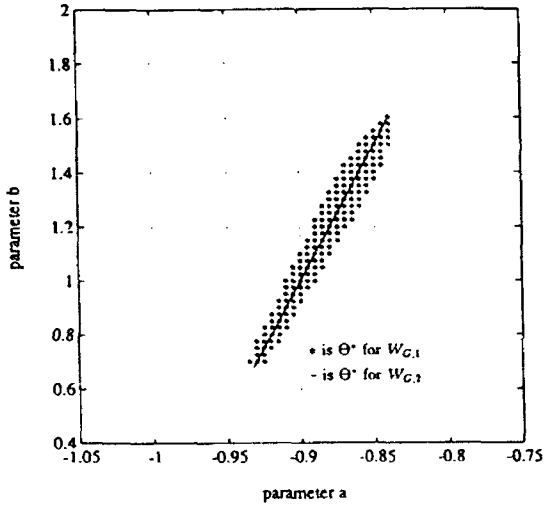

Fig. 4. Limit sets Θ^* for $W_{G,1}$ and $W_{G,2}$.

$T(z)$, then

$$\|T\|_{\mathcal{H}_\infty} = \sup_{|\omega| \le \pi} |T(e^{j\omega})| \tag{33}$$

$$= \sup_{\substack{\|x\|_{k2} \neq 0 \\ k \in \mathbb{N}}} \frac{\|Tx\|_{k2}}{\|x\|_{k2}} \tag{34}$$

$$= \sup \{\gamma : \|Tx\|_{k2} \le \gamma \|x\|_{k2}, \quad \forall \|x\|_{k2} < \infty, \forall k \in \mathbb{N}\}. \tag{35}$$

As a direct consequence, we also have

$$\sup_{\|T\|_{\mathcal{H}_\infty} \le \gamma} \|Tx\|_{k2} = \gamma \|x\|_{k2}. \tag{36}$$

To show that

$$\Theta^* \subseteq \Theta_k, \qquad \forall k \in \mathbb{N} \tag{37}$$

let $\theta^* \in \Theta^*$, i.e.,

$$G = \frac{B_{\theta^*}}{A_{\theta^*}}(1 + \Delta_G^* W_G) \tag{38}$$

with $\|\Delta_G^*\|_{\mathcal{H}_\infty} \le 1$. Note that θ^* and Δ_G^* must agree with the measured data, so

$$A_{\theta^*} y - B_{\theta^*} u = \Delta_G^* W_G B_{\theta^*} u. \tag{39}$$

Taking the l_2-norm, we have

$$\|A_{\theta^*} y - B_{\theta^*} u\|_{k2} = \|\Delta_G^* W_G B_{\theta^*} u\|_{k2}. \tag{40}$$

Since $\|\Delta_G^*\|_{\mathcal{H}_\infty} \le 1$, (36) implies that θ^* must satisfy

$$\|A_{\theta^*} y - B_{\theta^*} u\|_{k2} \le \|W_G B_{\theta^*} u\|_{k2}. \tag{41}$$

Therefore,

$$\theta^* \in \{\theta : \|A_\theta y - B_\theta u\|_{k2} \le \|W_G B_\theta u\|_{k2}\} = \Theta_k \tag{42}$$

for $\Theta^* \subseteq \Theta_k$. From this, it follows immediately that $\Theta^* \subseteq \Theta[N]$. □

A. Frequency-Domain Expressions

Define the asymptotic equation-error set as

$$\Theta_\infty \triangleq \lim_{k \to \infty} \Theta_k. \tag{43}$$

The limit set Θ^* and the asymptotic equation-error set Θ_∞ are expressed in the frequency domain in the following theorem.

Theorem 2:

i) The limit set has the following decomposition:

$$\Theta^* = \Theta^*_{stab} \cap \Theta^*_{freq} \tag{44}$$

where

$$\Theta^*_{stab} = \left\{\theta : \frac{A_\theta G - B_\theta}{W_G B_\theta} \text{ stable}\right\} \tag{45}$$

$$\Theta^*_{freq} = \{\theta : |A_\theta(e^{j\omega})G(e^{j\omega}) - B_\theta(e^{j\omega})| \le |W_G(e^{j\omega}) B_\theta(e^{j\omega})|, \forall |\omega| \le \pi\}. \tag{46}$$

ii) If $y = Gu$ and u has spectrum $S_{uu}(\omega)$, then

$$\Theta_\infty = \left\{\theta : \frac{1}{2\pi} \int_{-\pi}^{\pi} (|A_\theta G - B_\theta|^2 - |W_G B_\theta|^2) S_{uu}\, d\omega \le 0\right\}. \tag{47}$$

Proof: The decomposition of Θ^* follows directly from the definition of the $\mathcal{H}_\infty$ norm. The asymptotic set description is a direct application of the spectral expressions in (9). □

Theorem 1 states that $\Theta^* \subseteq \Theta_k$ for all k. It is clear from the frequency-domain expression for Θ^* that $\Theta^* \subseteq \Theta_\infty$ also because $\theta \in \Theta^*$ implies that the integrand in the frequency-domain expression for Θ_∞ is negative. Note also that the definition of Θ^* describes a parameter set via an $\mathcal{H}_\infty$ norm. By comparison, Θ_∞ is described via an $\mathcal{H}_2$ norm when u is white noise with $S_{uu}(\omega) = 1$, i.e.,

$$\Theta_\infty = \left\{\theta : \frac{\|A_\theta G - B_\theta\|_{\mathcal{H}_2}}{\|W_G B_\theta\|_{\mathcal{H}_2}} \le 1\right\}. \tag{48}$$

B. Use of Data Filtering

The effect of data filtering is to replace (y, u) with (Fy, Fu), where F is a filter with transfer function $F(z)$. Hence

$$\Theta_k = \{\theta: \| A_\theta Fy - B_\theta Fu\|_{k2} \le \| B_\theta W_G Fu\|_{k2}\}. \quad (49)$$

The effect of the filter is seen more clearly in the frequency-domain expression

$$\Theta_\infty = \left\{\theta: \frac{1}{2\pi}\int_{-\pi}^{\pi} (|A_\theta G - B|^2 - |W_G B_\theta|^2)|F|^2 S_{uu}\, d\omega \le 0\right\}. \quad (50)$$

The filter and the input spectrum form the frequency-dependent weight $|F(e^{j\omega})|^2 S_{uu}(\omega)$ which also appears in standard equation-error minimization methods [24].

C. Computing the Equation-Error Set

Ideally, it is desirable to compute $\Theta[N]$. This involves intersecting the N sets

$$\{\Theta_k: k = 1, \cdots, N\}.$$

We start with the following result which presents a convenient form for computing Θ_k.

Theorem 3: Define the following vectors whose elements are sequences:

$$\phi \triangleq \begin{bmatrix}\phi_y \\ \phi_u\end{bmatrix} \quad (51)$$

$$\phi_y \triangleq \begin{bmatrix}-z^{-1}y & \cdots & -z^{-n}y\end{bmatrix}^T \quad (52)$$

$$\phi_u \triangleq \begin{bmatrix}z^{-1}u & \cdots & z^{-m}u\end{bmatrix}^T. \quad (53)$$

Then,

i) Θ_k can be expressed in the quadratic form

$$\Theta_k = \{\theta: \theta^T\Gamma_k\theta - 2\beta_k^T\theta + \alpha_k \le 0\} \quad (54)$$

where $\alpha_k \in \mathbb{R}$, $\beta_k \in \mathbb{R}^p$, and $\Gamma_k \in \mathbb{R}^{p\times p}$ (with $p = m + n$) are given by

$$\alpha_k = \mathcal{E}_k(y^2) \quad (55)$$

$$\beta_k = \mathcal{E}_k(\phi y) \quad (56)$$

$$\Gamma_k = \mathcal{E}_k(\phi\phi^T) - \begin{bmatrix}0 & 0 \\ 0 & \mathcal{E}_k((W_G\phi_u)(W_G\phi_u)^T)\end{bmatrix}. \quad (57)$$

ii) Provided Γ_k^{-1} exists, another expression is

$$\Theta_k = \left\{\theta: (\theta - \hat\theta_k)^T\Gamma_k(\theta - \hat\theta_k) \le V_k\right\} \quad (58)$$

$$\hat\theta_k = \Gamma_k^{-1}\beta_k \quad (59)$$

$$V_k = \beta_k^T\Gamma_k^{-1}\beta_k - \alpha_k. \quad (60)$$

iii) All the eigenvalues of Γ_k are real and some of them can be negative. When $\Gamma_k > 0$, Θ_k is an ellipsoid in $\mathbb{R}^p$. When Γ_k is indefinite, Θ_k is an hyperboloid in $\mathbb{R}^p$.

Remarks: In part ii), the center of the set $\hat\theta_k$ is identical to the ordinary least-squares estimate when $W_G = 0$. This occurs only when nonparametric dynamics are neglected.

Proof: Using the definitions in the theorem, we have

$$A_\theta y - B_\theta u = y - \theta^T\phi \quad (61)$$

$$W_G B_\theta u = \theta^T W_G\begin{bmatrix}0 \\ \phi_u\end{bmatrix}. \quad (62)$$

Hence, substituting into (31), we have

$$\Theta_k = \left\{\theta \in \mathbb{R}^p: \| y - \theta^T\phi\|_{k2} \le \left\|\theta^T W_G\begin{bmatrix}0 \\ \phi_u\end{bmatrix}\right\|_{k2}\right\}. \quad (63)$$

Using (12), the quadratic form of Θ_k follows immediately, which proves part i).

Part ii) is obtained by direct substitution when Γ_k^{-1} exists.

To prove iii), observe that Γ_k can be expressed as follows:

$$\Gamma_k = \begin{bmatrix}\Gamma_{k,11} & \Gamma_{k,12} \\ \Gamma_{k,12}^T & \Gamma_{k,22}\end{bmatrix} \quad (64)$$

where

$$\Gamma_{k,11} = \mathcal{E}_k(\phi_y\phi_y^T) \quad (65)$$

$$\Gamma_{k,12} = \mathcal{E}_k(\phi_y\phi_u^T) \quad (66)$$

$$\Gamma_{k,22} = \mathcal{E}_k(\phi_u\phi_u^T - (W_G\phi_u)(W_G\phi_u)^T). \quad (67)$$

The $\Gamma_{k,22}$ matrix subblock can obviously cause Γ_k to have negative eigenvalues. The square roots of the eigenvalues of Γ_k^{-1} are the lengths of the semiaxes of the ellipsoid. Therefore, as Γ_k becomes singular, some directions of the ellipsoid become unbounded. A hyperboloid results when one or more eigenvalues of Γ_k become negative. □

Note that if the spectrum of u is concentrated at those frequencies where $|W_G(e^{j\omega})|$ is large, the $\Gamma_{k,22}$ matrix subblock can have negative eigenvalues. This tends to make Γ_k become indefinite, so that Θ_k becomes an hyperboloid. This will be illustrated in an example in the next section.

D. Example of Θ_k

The true system was selected, as in the previous example in Section III-B using the weight $W_{G,1}$ defined in (25). The input was a log-spaced sinesweep from 0.1 to 31 rad/s over 102.3 s, thus, $N = 1024$ data samples. Two filtered data sets were generated using eighth-order low-pass Butterworth filters; one with a bandpass of $\omega_f = 2$ rad/s, and the other with $\omega_f = 1$ rad/s.

Fig. 5 shows Θ_{1024} processed with the two data filters. An hyperboloid is obtained with $\omega_f = 2$ rad/s and an ellipsoid with $\omega_f = 1$ rad/s. (Note that only one branch of the hyperboloid is shown in the figure.) This confirms the earlier point that when u is concentrated at those frequencies where $|W_G(e^{j\omega})|$ is large, Θ_k can become unbounded. Points in the limit set Θ^* are shown and, as predicted by the theory, are all contained in the equation-error sets.

E. Computing Intersecting Ellipsoids

To compute $\Theta[N]$ requires computing the intersection of the sets $\{\Theta_k: k = 1, \cdots, N\}$. Since all the Θ_k are convex, it

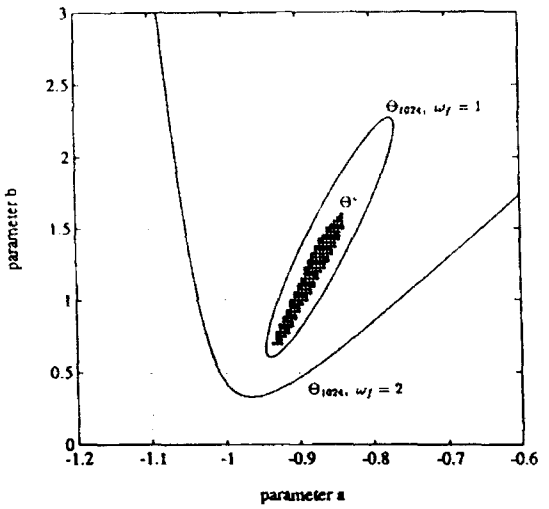

Fig. 5. Θ_{1024} for each data filter; points in limit set are also shown.

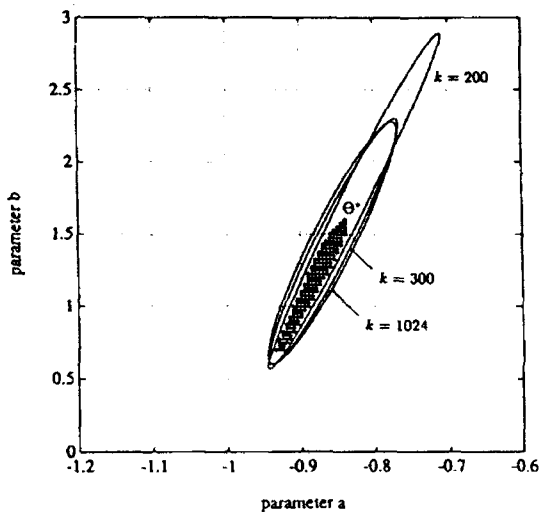

Fig. 6. The equation-error sets $\{\Theta_k\colon k = 200, 300, \cdots, 1024\}$ using the filtered data with $\omega_f = 1$ rad/s; Θ^* is also shown.

follows that $\Theta[N]$ is convex. In general, it is not, however, an ellipsoid. To see this, we plotted some of the bounding ellipsoid sets in Fig. 6. Specifically, it shows

$$\{\Theta_k\colon k = 200, 300, \cdots, 1024\}$$

corresponding to the previous example using the data filter with cutoff at 1 rad/s. Observe that the intersection of the sets produces a smaller (convex) set. Several approaches are possible. One approach is to compute the smallest volume ellipsoid that contains the intersection of the ellipsoids. This is discussed in [6] and [3].

F. Effect of Initial Conditions

As defined in Section II, the sequence Gu evaluated at time $t \in \mathbb{N}$ is defined by

$$(Gu)(t) = \sum_{\tau=1}^{t-1} g(\tau)u(t-\tau). \tag{68}$$

To account for initial conditions, let $\tilde{u}$ denote a bounded input applied for $t \le 0$. Thus, the system with initial conditions can be expressed as

$$y = Gu + \tilde{y} \tag{69}$$

with

$$\tilde{y}(t) = \sum_{\tau=t}^{\infty} g(\tau)\tilde{u}(t-\tau), \qquad \forall t \in \mathbb{N}. \tag{70}$$

If G is stable or is in a stabilizing feedback, then $\tilde{y}(t) \to 0$ exponentially as $t \to \infty$. Thus, the effect of initial conditions dies out exponentially fast, or slow, depending on the slowest modes in G or the closed-loop system. Hence, for sufficiently large N, we have $\Theta_N \approx \Theta_\infty$. More precisely, for each $\theta \in \Theta_\infty$,

$$\lim_{N\to\infty} \inf_{\hat{\theta}_N \in \Theta_N} \|\theta - \hat{\theta}_N\| = 0 \tag{71}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^p$. In words, the estimator will eventually report possible parameter values that are close to the asymptotic set, and hence, asymptotically bound the limit set Θ^* as the data length N increases.

Another way to account for the effect of initial condition is to assume bounds on $\tilde{u}$ and the tail of g

$$\sum_{\tau=t}^{\infty} |g(\tau)| \le \kappa_1 \tag{72}$$

$$|\tilde{u}(t)| \le \kappa_2, \qquad t \le 0. \tag{73}$$

Then $|\tilde{y}(t)| \le \kappa_1\kappa_2$, and it can be treated as a bounded disturbance in (69), see e.g., [30].

G. Other Forms of Nonparametric Uncertainty

The equation-error sets we have developed so far assume a multiplicative form of nonparametric uncertainty. This is not a necessary restriction as they could also have been developed for other forms. The requisite modifications are shown below for some other typical forms.

Theorem 4:

i) *Multiplicative*: If

$$G = \frac{B_\theta}{A_\theta}(1 + \Delta_G W_G), \qquad \|\Delta_G\|_{\mathcal{H}_\infty} \le 1 \tag{74}$$

then

$$\Theta_k = \{\theta\colon \|A_\theta y - B_\theta u\|_{k2} \le \|W_G B_\theta u\|_{k2}\}. \tag{75}$$

ii) *Additive*: If

$$G = \frac{B_\theta}{A_\theta} + \Delta_G W_G, \qquad \|\Delta_G\|_{\mathcal{H}_\infty} \le 1 \tag{76}$$

then

$$\Theta_k = \{\theta\colon \|A_\theta y - B_\theta u\|_{k2} \le \|W_G A_\theta u\|_{k2}\}. \tag{77}$$

iii) *Inverse Multiplicative*: If

$$G = \frac{B_\theta}{A_\theta}\left(\frac{1}{1 + \Delta_G W_G}\right), \qquad \|\Delta_G\|_{\mathcal{H}_\infty} \le 1 \tag{78}$$

then

$$\Theta_k = \{\theta\colon \|A_\theta y - B_\theta u\|_{k2} \le \|W_G A_\theta y\|_{k2}\}. \tag{79}$$

iv) *Feedback*: If

$$G = \frac{B_\theta}{A_\theta}\left(\frac{1}{1 + \Delta_G W_G \frac{B_\theta}{A_\theta}}\right), \qquad \|\Delta_G\|_{\mathcal{H}_\infty} \le 1 \quad (80)$$

then

$$\Theta_k = \{\theta : \| A_\theta y - B_\theta u\|_{k2} \le \| W_G B_\theta y\|_{k2}\}. \quad (81)$$

v) *Coprime Factored (Coupled)*: If

$$G = \frac{B_\theta + \Delta_B W_B}{A_\theta + \Delta_A W_A}, \qquad \left\|\begin{bmatrix}\Delta_B \\ \Delta_A\end{bmatrix}\right\|_{\mathcal{H}_\infty} \le 1 \quad (82)$$

then

$$\Theta_k = \left\{\theta : \| A_\theta y - B_\theta u\|_{k2} \le \left\|\begin{bmatrix} W_B u \\ W_A y\end{bmatrix}\right\|_{k2}\right\}. \quad (83)$$

vi) *Coprime Factored (Uncoupled)*: If

$$G = \frac{B_\theta + \Delta_B W_B}{A_\theta + \Delta_A W_A}, \qquad \|\Delta_B\|_{\mathcal{H}_\infty} \le 1, \ \|\Delta_A\|_{\mathcal{H}_\infty} \le 1 \quad (84)$$

then

$$\Theta_k = \{\theta : \| A_\theta y - B_\theta u\|_{k2} \le \| W_B u\|_{k2} + \| W_A y\|_{k2}\}. \quad (85)$$

vii) *All the above set estimates* Θ_k have the property that

$$\Theta^* \subseteq \Theta[N] \subseteq \Theta_k. \quad (86)$$

Proof: The proof of the property $\Theta^* \subseteq \Theta_k$ for all the cases above is similar to the proof for Theorem 1. We will show it for case vi) only. Let $\theta^* \in \Theta^*$, i.e.,

$$\frac{B_{\theta^*} + \Delta_B^* W_B}{A_{\theta^*} + \Delta_A^* W_A} \quad (87)$$

with

$$\|\Delta_B^*\|_{\mathcal{H}_\infty} \le 1 \text{ and } \|\Delta_A^*\|_{\mathcal{H}_\infty} \le 1. \quad (88)$$

Since θ^*, Δ_A^*, and Δ_B^* must agree with the measured data

$$A_{\theta^*} y - B_{\theta^*} u = \Delta_B^* W_B u - \Delta_A^* W_A y. \quad (89)$$

Now take the l_2-norm and apply the triangle inequality with (88), θ^* must also satisfy

$$\| A_{\theta^*} y - B_{\theta^*} u\|_{k2} \le \| W_B u\|_{k2} + \| W_A y\|_{k2}. \quad (90)$$

Therefore,

$$\theta^* \in \{\theta : \| A_\theta y - B_\theta u\|_{k2} \le \| W_B u\|_{k2} + \| W_A y\|_{k2}\} = \Theta_k \quad (91)$$

and $\Theta^* \subseteq \Theta_k$. □

From these forms it is straightforward to generate the corresponding quadratic forms for computing the sets. In those cases, when the right-hand side of any of the above inequalities does not depend on the parameter θ, the center of the parametric set is the usual least-squares estimate, e.g., [32].

V. Equation-Error Set Estimation with Disturbances

There are many ways to characterize the disturbance environment both in terms of the location and the type of disturbance. To simplify the discussion, we assume that the disturbance is located additively at the output, as given by (16)

$$y = Gu + v.$$

The most common type is the stochastic disturbance which we consider in this section. Deterministic "worst-case" types of disturbances are discussed briefly in Section VI.

A. Stochastic Additive Disturbance

Suppose that the disturbance v is a zero-mean quasi-stationary sequence in the set

$$\mathcal{V} = \{v : S_{vv}(\omega) \le \sigma^2 | W_H(e^{j\omega})|^2, \ S_{vu}(\omega) = 0, \forall |\omega| \le \pi\} \quad (92)$$

where $W_H(z)$ is a stable and stably invertible transfer function. Equivalently, we can think of v as the output of a stable uncertain linear-time invariant system H with a white-noise input e. Hence,

$$v = He \quad (93)$$

where H is in the set of linear-time-invariant systems $\mathcal{H}$ and e is in the set of stochastic sequences $\mathcal{W}_{\text{stoch}}$ defined as follows:

$$\mathcal{H} \triangleq \{\Delta_H W_H \text{ stable} : \|\Delta_H\|_{\mathcal{H}_\infty} \le 1\} \quad (94)$$

$$\mathcal{W}_{\text{stoch}} \triangleq \{\text{white noise } e : S_{ee}(\omega) = \sigma_e^2 \le \sigma^2, \ S_{eu}(\omega) = 0, \ \forall |\omega| \le \pi, \ \text{bounded fourth moment}\}. \quad (95)$$

The disturbance set then becomes

$$\mathcal{V} = \{v = He : H \in \mathcal{H}, \ e \in \mathcal{W}_{\text{stoch}}\}. \quad (96)$$

Assuming that W_H and σ are known, the disturbance set defined above is otherwise parameter-free. One can compare this set description to $\mathcal{G}$ which contains the parametric transfer function $G_\theta(z)$. As it is, the disturbance set is perfectly adequate for describing a sensor noise. However, in the case of a general disturbance reflected to the output, the set merely serves to provide an upper bound. For small disturbances this is adequate, but the set is potentially conservative otherwise. For a more complete discussion on this matter, see [19].

We now have the following.

Theorem 5: Suppose that the true plant which generated $\{y, u : t = 1, \cdots, N\}$ has the structure described above. Then

i)

$$\Theta_N \to \Theta_\infty \qquad \text{w.p. 1 as } N \to \infty. \quad (97)$$

ii)

$$\Theta^* \subseteq \Theta_\infty \quad (98)$$

where the equation-error sets are now defined as follows:

$$\Theta_N \triangleq \left\{\theta: \hat{\mathcal{E}}_N\left(\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\right) \le \hat{\mathcal{E}}_N\left(\left(W_H^{-1}W_G B_\theta u\right)^2\right) + \sigma^2(1 + \theta_A^T\theta_A)\right\} \quad (99)$$

$$\Theta_\infty \triangleq \left\{\theta: \bar{\mathcal{E}}\left(\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\right) \le \bar{\mathcal{E}}\left(\left(W_H^{-1}W_G B_\theta u\right)^2\right) + \sigma^2(1 + \theta_A^T\theta_A)\right\} \quad (100)$$

with

$$\theta_A \triangleq [a_1 \cdots a_n]^T. \quad (101)$$

iii) In the frequency domain

$$\Theta_\infty = \left\{\theta: \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_\theta(\omega)}{|W_H(e^{j\omega})|^2}\, d\omega \le 0\right\} \quad (102)$$

with

$$f_\theta(\omega) = (|A_\theta G - B_\theta|^2 - |W_G B_\theta|^2)S_{uu}(\omega) + |A_\theta|^2(|H|^2 S_{ee}(\omega) - |W_H|^2\sigma^2). \quad (103)$$

Observe that both the finite-data set Θ_N, as well as the infinite-data set Θ_∞, depend on the noise intensity σ and the disturbance weighting transfer function W_H, whose inverse acts as a data filter. The theorem is analogous to the many prediction-error based parameter estimators in the sense that for a sufficiently long data length N, the estimate is equal to the true value with high probability [24]. In our case, the finite-data set Θ_N will contain Θ^* with high probability. Part i) of the theorem means that for each $\theta \in \Theta_\infty$, there is a $\hat{\theta}_N \in \Theta_N$ close to it as N increases. More precisely,

$$\inf_{\hat{\theta}_N \in \Theta_N} \|\theta - \hat{\theta}_N\| \to 0 \quad \text{w.p. 1 as } N \to \infty \quad (104)$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^p$.

The integrand in the frequency-domain expression for Θ_∞ is always negative provided that for all $|\omega| \le \pi$

$$\left|G - \frac{B_\theta}{A_\theta}\right|^2 - \left|W_G\frac{B_\theta}{A_\theta}\right|^2 \le \frac{1}{S_{uu}}(|W_H|^2\sigma^2 - |H|^2 S_{ee}). \quad (105)$$

We can now see the usual effects of signal-to-noise ratio. As the noise power σ^2 increases, the "volume" in Θ_∞ will increase. Conversely, if $S_{uu}(\omega)$ is large at many frequencies, Θ_∞ will shrink. In addition, in the frequency ranges where $|W_H(e^{j\omega})| \gg H(e^{j\omega})$, an indication of poor prior information, very large-input power at these frequencies is required to keep Θ_∞ small.

Proof: Under the assumptions, the true system can be expressed as

$$y = \frac{B_\theta}{A_\theta}(1 + \Delta_G W_G)u + \Delta_H W_H e \quad (106)$$

for some $\|\Delta_G\|_{\mathcal{H}_\infty} \le 1$, $\|\Delta_G\|_{\mathcal{H}_\infty} \le 1$, and $e \in \mathcal{W}_{\text{stoch}}$. Rearranging terms and filtering by W_H^{-1} gives

$$W_H^{-1}(A_\theta y - B_\theta u) = \Delta_G W_H^{-1}W_G B_\theta u + \Delta_H A_\theta e. \quad (107)$$

Squaring both sides and taking autocorrelation at $\tau = 0$, we get

$$\bar{\mathcal{E}}\left(\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\right) = \bar{\mathcal{E}}\left(\left(\Delta_G W_H^{-1}W_G B_\theta u\right)^2\right) + \bar{\mathcal{E}}\left(\left(\Delta_H A_\theta e\right)^2\right) \quad (108)$$

where the cross terms (between e and u) are zero because e and u are independent. Now take the supremum of the right-hand side to obtain the infinite-data parameter set

$$\Theta_\infty = \left\{\theta: \bar{\mathcal{E}}\left(\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\right) \le \sup_{\Delta_G, \Delta_H, e}\left[\bar{\mathcal{E}}\left(\left(\Delta_G W_H^{-1}W_G B_\theta u\right)^2\right) + \bar{\mathcal{E}}\left(\left(\Delta_H A_\theta e\right)^2\right)\right]\right\}. \quad (109)$$

To evaluate the right-hand side above, we now use the assumptions $\|\Delta_G\|_{\mathcal{H}_\infty} \le 1$, $\|\Delta_H\|_{\mathcal{H}_\infty} \le 1$, and $e \in \mathcal{W}_{\text{stoch}}$ to obtain

$$\sup_{\|\Delta_G\|_{\mathcal{H}_\infty} \le 1} \bar{\mathcal{E}}\left(\left(\Delta_G W_H^{-1}W_G B_\theta u\right)^2\right) = \bar{\mathcal{E}}\left(\left(W_H^{-1}W_G B_\theta u\right)^2\right) \quad (110)$$

$$\sup_{e \in \mathcal{W}_{\text{stoch}}} \sup_{\|\Delta_H\|_{\mathcal{H}_\infty} \le 1} \bar{\mathcal{E}}\left(\left(\Delta_H A_\theta e\right)^2\right) = \sup_{e \in \mathcal{W}_{\text{stoch}}} \bar{\mathcal{E}}\left(\left(A_\theta e\right)^2\right) \quad (111)$$

$$= \sigma^2 \|A_\theta\|_{\mathcal{H}_2}^2 \quad (112)$$

$$= \sigma^2\left(1 + \sum_{k=1}^{n} a_k^2\right) \quad (113)$$

$$= \sigma^2(1 + \theta_A^T\theta_A). \quad (114)$$

This yields the set Θ_∞ as defined in the theorem.

Observe that Θ_N has precisely the same form as Θ_∞ except that the operator $\bar{\mathcal{E}}(\cdot)$ is replaced everywhere with the sample mean $\hat{\mathcal{E}}_N(\cdot)$. To show (97), recall from [24, pp. 34–35] that if the stochastic part of x can be described as filtered white noise, then the spectrum of an observed single realization of x, computed as for a deterministic signal, coincides, with probability 1, with that of the process, i.e.,

$$\lim_{N\to\infty} \hat{\mathcal{E}}_N(x^2) \to \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathcal{E}\left(x(t)^2\right) = \bar{\mathcal{E}}(x^2). \quad (115)$$

The conditions for this convergence are that x is a quasi-stationary sequence and the white noise has bounded fourth moment. Note that since u and y are assumed to have finite power, $W_H^{-1}(A_\theta y - B_\theta u)$ and $W_H^{-1}W_G B_\theta u$ are quasi-stationary. Thus, the convergence in (97) holds.

To show that $\Theta^* \subseteq \Theta_\infty$, we use the frequency-domain expressions in iii). Observe that the frequency-domain expression for Θ_∞ can be obtained by substituting $y = Gu +$

He in (100) to get

$$\Theta_\infty = \left\{\theta: \bar{\mathcal{E}}\left(\left[W_H^{-1}(A_\theta G - B_\theta)u + W_H^{-1}A_\theta He\right]^2\right) \le \bar{\mathcal{E}}\left(\left(W_H^{-1}W_G B_\theta u\right)^2\right) + \sigma^2 \| A_\theta \|_{\mathcal{H}_2}^2\right\}. \quad (116)$$

Now use the fact that u and e are independent to simplify, then apply Parseval's theorem. In the frequency-domain expression, the assumption $H \in \mathcal{H}'$ means that

$$|H(e^{j\omega})|^2 S_{ee}(\omega) - |W_H(e^{j\omega})|^2\sigma^2 \le 0, \qquad \forall\omega \quad (117)$$

and $\theta \in \Theta^*$ means that

$$|A_\theta G - B_\theta|^2 - |W_G B_\theta|^2 \le 0, \qquad \forall\omega. \quad (118)$$

Thus, $\theta \in \Theta^*$ guarantees that $f_\theta(\omega)$ is negative for all frequencies, and hence, $\Theta^* \subseteq \Theta_\infty$. □

1) Example of Bias Estimation: As an illustrative example, consider estimating a constant in noise

$$y(t) = b_0 + e(t). \quad (119)$$

In this case, $W_G(z) = 0$ to reflect the absence of nonparametric uncertainty, and $H(z) = 1$. In addition, $W_H(z) = 1$, and $H \in \mathcal{H}'$ is satisfied. If $e \in \mathcal{W}_{\text{stoch}}$, then the set estimate for b_0 is

$$\Theta_N = \left\{b: (b - \hat{b})^2 \le \sigma^2 - \mathcal{E}_N\left((y - \hat{b})^2\right)\right\} \quad (120)$$

where $\hat{b} = \mathcal{E}_N(y)$. For large N, the right-hand side behaves as $\sigma^2 - \sigma_0^2$, where σ_0^2 is the true noise variance. Note that the limit set Θ^*, in this case, is the point b_0. Since $\hat{b} \to b_0$ as $N \to \infty$, we see that $\Theta^* \subseteq \Theta_\infty$ as stated in the theorem. Furthermore, as the bounding variance σ approaches σ_0, the set Θ_∞ becomes a point. Observe that Θ_∞ does not shrink to a point when there is nonparametric uncertainty, i.e., $W_G(z) \neq 0$.

B. Computing the Equation-Error Set

For computing Θ_N, we have the following result

Theorem 6: As in Theorem 3, define the vector sequences ϕ, ϕ_y, and ϕ_u. Then:

i) Θ_N can be expressed in the quadratic form

$$\Theta_N = \{\theta: \theta^T\Gamma_N\theta - 2\beta_N^T\theta + \alpha_N \le 0\} \quad (121)$$

where $\alpha_N \in \mathbb{R}$, $\beta_N \in \mathbb{R}^p$, and $\Gamma_N \in \mathbb{R}^{p\times p}$ are given by

$$\alpha_N = \mathcal{E}_N\left(\left(W_H^{-1}y\right)^2\right) - \sigma^2 \quad (122)$$

$$\beta_N = \mathcal{E}_N\left(\left(W_H^{-1}\phi\right)\left(W_H^{-1}y\right)\right) \quad (123)$$

$$\Gamma_N = \mathcal{E}_N\left(\left(W_H^{-1}\phi\right)\left(W_H^{-1}\phi\right)^T\right) - \begin{bmatrix}\sigma^2 I_n & 0\\ 0 & \mathcal{E}_N\left(\left(W_H^{-1}W_G\phi_u\right)\left(W_H^{-1}W_G\phi_u\right)^T\right)\end{bmatrix}. \quad (124)$$

ii) Provided Γ_N^{-1} exists, another expression is

$$\Theta_N = \left\{\theta: (\theta - \hat{\theta}_N)^T\Gamma_N(\theta - \theta_N) \le V_N\right\} \quad (125)$$

$$\hat{\theta}_N = \Gamma_N^{-1}\beta_N \quad (126)$$

$$V_N = \beta_N^T\Gamma_N^{-1}\beta_N - \alpha_N. \quad (127)$$

iii) When $\Gamma_N > 0$, Θ_N is an ellipsoid in $\mathbb{R}^p$ and when Γ_N is indefinite, Θ_N is a hyperboloid in $\mathbb{R}^p$.

The proof of the above proceeds along the same lines as that of Theorem 3, and is omitted.

The infinite-data parameter set estimate can also be expressed in a form identical to that for the finite-data set

$$\Theta_\infty = \{\theta: \alpha - 2\beta^T\theta + \theta^T\Gamma\theta \le 0\} \quad (128)$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^p$, and $\Gamma \in \mathbb{R}^{p\times p}$ are given by

$$\alpha = \bar{\mathcal{E}}\left(\left(W_H^{-1}y\right)^2\right) - \sigma^2 \quad (129)$$

$$\beta = \bar{\mathcal{E}}\left(\left(W_H^{-1}\phi\right)\left(W_H^{-1}y\right)\right) \quad (130)$$

$$\Gamma = \bar{\mathcal{E}}\left(\left(W_H^{-1}\phi\right)\left(W_H^{-1}\phi\right)^T\right) - \begin{bmatrix}\sigma^2 I_n & 0\\ 0 & \bar{\mathcal{E}}\left(\left(W_H^{-1}W_G\phi_u\right)\left(W_H^{-1}W_G\phi_u\right)^T\right)\end{bmatrix}. \quad (131)$$

C. Example of Θ_N with Disturbance

The example system is as before with G given by (23), and W_G given by (25). The disturbance dynamics is

$$H(z) = \frac{0.1}{z - 0.9} \quad (132)$$

and the disturbance weight is

$$W_H(z) = \frac{1}{\delta_H}H(z) \quad (133)$$

where $\delta_H \in (0, 1)$ is a parameter chosen by the user.

The disturbance v is simulated as the output of H driven by e, a sequence of independently distributed Gaussian variables with zero mean and variance σ^2. Three series of experiments are carried out to study the effects of noise power (choice of σ), mismatch between H and W_H (choice of δ_H), and length of data record (choice of N). In the first two experiments, the input u is a linearly-spaced sinesweep from 0.01 to 0.5 rad/s over 102.3 s, giving $N = 1024$ data samples. In the third experiment, N is varied.

To study the effects of noise power, σ is varied in this experiment. As suggested by Theorem 5, the parameter set estimate should expand as σ increases. This is supported by Fig. 7, where Θ_N is plotted for $\sigma = 0.1, 0.2$, and 0.4. Note that in all cases shown here, $\Theta^* \subseteq \Theta_N$.

In Fig. 8, the value of δ_H is varied from 0.6 to 1.0. Again, as suggested by Theorem 5, as the mismatch between H and W_H becomes larger, i.e., $|\delta_H|$ becomes smaller, Θ_N grows.

The effects of different data record lengths are studied in the last experiment. For the cases of $N = 1024$ and 2048 with $\sigma = 0.5$, and $\delta_H = 1.0$, Θ^* is not in Θ_N. This is still in agreement with our results because in the stochastic disturbance case Θ^* is only guaranted to be in Θ_N as N tends to infinity. As shown in Fig. 9, Θ^* is in Θ_N for $N = 4096$.

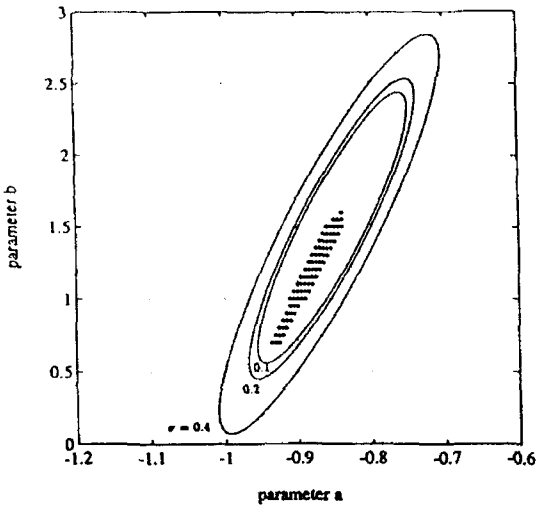

Fig. 7. Θ_N for different values of σ ($N = 1024$, $\delta_H = 0.8$).

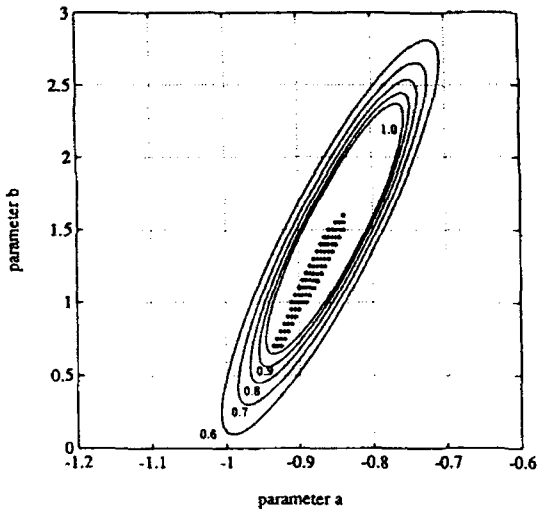

Fig. 8. Θ_N for different values of δ_H ($\sigma = 0.2$, $N = 1024$).

VI. Some Extensions

In this section, we first consider the extension of our results for the equation-error set estimates to the output-error set. We then consider disturbances which are deterministic in nature rather than stochastic, as considered in the previous section.

A. Disturbance-Free Output-Error Set Estimation

The results obtained for the equation-error set in Section IV can be repeated *mutatis mutandis* for the output-error set, but for the notable exception of forming a quadratic set for computational purposes none exists for output-error identification [24].

Theorem 7: Suppose the measured data $\{y, u: t = 1, \cdots, N\}$ is generated from $y = Gu$ with $G \in \mathcal{G}$. Then the following holds:

$$\Theta^* \subseteq \Theta^{oe}[N] \subseteq \Theta_k^{oe}, \qquad \forall k \in [1, N], \forall N \in \mathbb{N} \quad (134)$$

where $\Theta^{oe}[N]$ and Θ_k^{oe} are the output-error set estimates

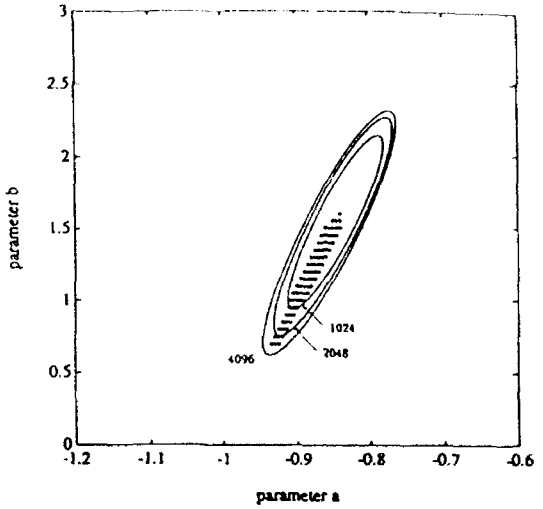

Fig. 9. Θ_N for different values of N ($\sigma = 0.5$, $\delta_H = 1.0$).

given by

$$\Theta_k^{oe} = \left\{\theta: \left\| y - \frac{B_\theta}{A_\theta} u \right\|_{k2} \le \left\| W_G \frac{B_\theta}{A_\theta} u \right\|_{k2}\right\} \quad (135)$$

$$\Theta^{oe}[N] = \bigcap_{k=1}^{N} \Theta_k^{oe}. \quad (136)$$

Remark: We refer to Θ_k^{oe} and $\Theta^{oe}[N]$ as *output-error parameter sets* because the output-error term $y - (B_\theta / A_\theta)u$ appears in their descriptions.

Proof: The proof of $\Theta^* \subseteq \Theta^{oe}[N] \subseteq \Theta_k^{oe}$ is identical to the one for Theorem 1. □

The sets Θ_k and Θ_k^{oe} are both worst-case estimates, both contain Θ^*, but they are not necessarily the same sets for identical input sequences. Another major difference is that both sides of the inequality in Θ_k are affine in θ, whereas in Θ_k^{oe} they are linear fractional in θ. The former property makes it very easy to compute Θ_k, as has been shown, whereas the latter makes it difficult to compute the output-error sets, as usual.

B. Deterministic Additive Disturbances

So far, we have only considered stochastic disturbances. We now briefly examine the effect of deterministic disturbances.

Suppose, as before, that the true system is

$$y = Gu + He \quad (137)$$

with $G \in \mathcal{G}$ and $H \in \mathcal{H}$ as previously described. We now consider the following deterministic set which describes quasi-stationary sequences with bounded spectra:

$$\mathcal{W}_{spec} = \{e(t): S_{ee}(\omega) \le \sigma_{spec}^2, \forall |\omega| \le \pi\}. \quad (138)$$

We then obtain the following.

Theorem 8: If $e \in \mathcal{W}_{spec}$, then

$$\Theta_k = \left\{\theta: \sqrt{\mathcal{E}_k\left(\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\right)} \le \sqrt{\mathcal{E}_k\left(\left(W_H^{-1} W_G B_\theta u\right)^2\right)} + \sigma_{spec}\sqrt{1 + \theta_A^T \theta_A}\right\} \quad (139)$$

and

$$\Theta^* \subseteq \lim_{k \to \infty} \Theta_k. \tag{140}$$

Proof: The proof of (140) proceeds the same as Theorems 1 and 4. Let $\theta^* \in \Theta^*$, then

$$W_H^{-1}(A_{\theta^*} y - B_{\theta^*} u) = \Delta_G^* W_H^{-1} W_G B_{\theta^*} u + \Delta_H A_{\theta^*} e. \tag{141}$$

After squaring both sides and taking the sample averages, the Schwarz's inequality, $\|\Delta_G^*\|_{\mathcal{H}_\infty} \le 1$, and $\|\Delta_H\|_{\mathcal{H}_\infty} \le 1$ are applied to obtain

$$\sqrt{\bar{\mathcal{E}}_k\left(\left[W_H^{-1}(A_{\theta^*} y - B_{\theta^*} u)\right]^2\right)} \le \sqrt{\bar{\mathcal{E}}_k\left(\left(W_H^{-1} W_G B_{\theta^*} u\right)^2\right)} + \sqrt{\bar{\mathcal{E}}_k\left(\left(A_{\theta^*} e\right)^2\right)}. \tag{142}$$

Now let $k \to \infty$, we have

$$\theta^* \in \left\{\theta : \sqrt{\bar{\mathcal{E}}\left(\left[W_H^{-1}(A_\theta y - B_\theta u)\right]^2\right)} \le \sqrt{\bar{\mathcal{E}}\left(\left(W_H^{-1} W_G B_\theta u\right)^2\right)} + \sigma_{\text{spec}} \sqrt{1 + \theta_A^T \theta_A} \right\} \tag{143}$$

and $\Theta^* \subseteq \lim_{k \to \infty} \Theta_k$. □

In both cases of stochastic and deterministic disturbances, the limit set Θ^* is contained in the set estimate as the data length tends to infinity. However, in the deterministic case here, the probabilistic convergence need not be considered. The reason that both cases can be handled in the same way is because a common framework is used for deterministic and stochastic signals, [see (5) and (6)]. Note that instead of using $\mathcal{W}_{\text{spec}}$ to describe the deterministic disturbance e, we can also use

$$\mathcal{W}_{\text{rms}} = \left\{ e(t) : \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} e(t)^2 \le \sigma_{\text{rms}}^2 \right\} \tag{144}$$

to describe e and obtain results similar to Theorem 8.

VII. Concluding Remarks

The set-membership approach to system identification starts with the assumption that the underlying true system which generated the measured data is in a known set characterized by some unknown parameters and unknown but bounded nonparametric dynamics. We then derived set estimates for these unknown parameters. In the disturbance-free case, the set estimate has the property that it always contains the limit set. In the presence of stochastic disturbances, the set estimate is shown to have the property that it contains the limit set with probability one as the data length tends to infinity.

The set estimates derived in this paper also have some nice properties for computation. For the equation-error estimates, the set expressions are quadratic in the parameters. Thus, the set estimates are either ellipsoids or hyperboloids in the parameter space. Furthermore, these sets are easily obtained by computing averages of the filtered input-output data. However, when the output-error form is used in the set estimate, these nice properties are lost, which is typical with output-error identification.

The next step is to use these set estimates with a robust on-line control design procedure. One approach would be to bury the parameter uncertainty in another nonparametric uncertainty by finding an overbounding frequency-dependent weighting function. This is a potentially very conservative approach. Alternatively, the minimax approach in [22] and [21] presents a robust control-design procedure to handle the specific type of parameter uncertainty as represented by the ellipsoidal sets. This is a current topic of our research.

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Robert L. Kosut (S'67–M'69–M'75–SM'90–F'92) was born in the Bronx, NY, on May 28, 1942. He received the B.S. degree in electrical engineering from the City College of New York in 1965, and the M.S. and Ph.D. degrees in electrical engineering from the University of Pennsylvania, Philadelphia, in 1967 and 1969, respectively.

He is currently the Manager of the Research Group at Integrated Systems, Inc., Santa Clara, CA, and is also a Consulting Professor in the Department of Electrical Engineering at Stanford University, Stanford, CA. He has held appointments as a Visiting Fellow in the Department of Systems Engineering at the Australian National University, Canberra, Australia, and in the Department of Automatic Control at Lund Institute of Technology, Lund, Sweden. He is a coauthor of the book, *Stability of Adaptive Systems: Passivity and Averaging Analysis* (Cambridge, MA: M.I.T. Press, 1986).

Dr. Kosut was an Associate Editor for the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and is currently a Guest Editor for a Special Issue on System Identification for Robust Control Design. He is also on the Editorial Board of the *International Journal of Adaptive Control and Signal Processing*.

Ming K. Lau (S'78–M'81–S'89) received the B.S.E.E. degree from the University of Missouri at Rolla, in 1980, and the M.S.E.E. and Ph.D. degrees from Stanford University, Stanford, CA, in 1982 and 1992, respectively.

Since 1981, he has been a Member of the Technical Staff at Sandia National Laboratories, Albuquerque, NM. His assignments at Sandia included algorithm designs for terrain-aided navigation systems for helicopters and aircrafts using radar-altimeter and synthetic-aperture-radar measurements with stored terrain elevation data. In 1988 he returned to Stanford to work on the Ph.D. degree under Sandia's doctoral study program. His thesis research is in the areas of set-membership identification and control design.

Stephen P. Boyd (S'82–M'85) received the A.B. degree in mathematics, suma cum laude, from Harvard University, Cambridge, MA, in 1980, and the Ph.D. degree in EECS from the University of California, Berkeley, in 1985.

In 1985 he joined the Department of Electrical Engineering, Stanford University, Stanford, CA, where he is currently Associate Professor.

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Robust Control Design for Ellipsoidal Plant Set

MING K. LAU*† STEPHEN BOYD* ROBERT L. KOSUT‡ GENE F. FRANKLIN*

Abstract This paper presents a control design method for continuous-time plants whose uncertain parameters in the output matrix are only known to lie in an ellipsoidal set. The desired control is chosen to minimize the maximum linear quadratic regulator (LQR) cost from all the plants with parameters in the given set. Although no particular form is assumed for the minimax control, it turns out that it is the LQR control for one of the plants in the set, the worst-case plant. By defining an appropriate mapping, which maps an element from the given ellipsoidal set to an element of the same set, the existence of this worst-case plant is proved. A simple heuristic algorithm used to compute the worst-case plant is also given.

1 Introduction

A problem of great interest in control theory is the design of a controller which can guarantee some level of performance in the presence of plant parameter uncertainty. Kharitonov's theorem provides a necessary and sufficient analysis test for determining the robust stability of polynomials with perturbed coefficients, however, there are few results that exploit Kharitonov's theorem for synthesizing robust controllers, *e.g.*, [7] and [12]. Another approach to this problem is to define a set of nominal values of the uncertain parameters and consider deviations from these nominal values. A comprehensive survey of the different parameter space methods for robust control design, as opposed to frequency domain methods, can be found in [23].

The technique of solving control problems as minimax optimization problems is the basis of the so-called "$\mathbf{H}_\infty$ optimal control theory." In the standard $\mathbf{H}_\infty$ problem, the control input is chosen to minimize the norm of the output and the exogenous input is chosen to maximize it [2]. Along this line, the structured singular value (μ) synthesis method is used to find controllers which minimize a $\mathbf{H}_\infty$ objective subject to plant perturbations, *e.g.*, see [8], [9], and references therein. In [20], a game theoretic approach is used, where the control, restricted to a function of the measurement history, plays against adversaries composed of the process and measurement disturbances, and the initial states. Another example of solving control problems as minimax problems is [18], which presents a controller design method to minimize the weighted sum of the maximum linear quadratic gaussian (LQG) performance objectives over a set of worst plant parameter changes.

The approach of using set-membership to describe plant parameter uncertainty has gained popularity in recent years, *e.g.*, [14], [16], [26], [3], [17], and references therein. This approach of parameter identification is originated from early works of [22] and [5], where the set of possible system states compatible with the observations is shown to be an ellipsoid. Motivated by ellipsoidal bounds on plant parameters, we pose the following robust control problem: given that the unknown parameters in the output matrix of the plant are known to lie in an ellipsoid, find the control which minimizes the maximum LQR cost from all plants with parameters in the given set. Viewed in terms of game theory, the control and plant uncertainty are strategies employed by opposing players in a game, where the control is chosen to minimize the LQR cost and the plant uncertainty is chosen to maximize it. As a special case of our problem, finding the finite-horizon control for a discrete finite-impulse response (FIR) plant, was solved in [15]. In that case, it was shown that the minimization is a convex optimization problem. In this paper, we are generalizing the robust control design problem to find the infinite-horizon controls for continuous plants.

The assumption that the output matrix in the plant description contains all the uncertainty deserves further discussion. First, this is a natural extension of the discrete FIR finite-horizon problem solved in [15]. In the discrete case, FIR model sets can be identified from input-output data of a plant, *i.e.*, the coefficients of the FIR model are identified to belong to a set. This is particularly attractive when a bounded noise model, often a more realistic assumption than a statistical noise model, is used in the identification [19]. In the continuous case, Laguerre models can be used so that the identification is reduced to estimating the Laguerre coefficients [25]. Uncertainty in the Laguerre coefficients can then be described by set membership of the output matrix. Second, by limiting uncertain parameters to the output matrix, we simplify the analysis and gain more insights into the nature of the solution.

The paper is organized as follows, after stating the

*Information Systems Lab, Stanford University, Stanford, CA 94305.

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‡Integrated Systems Inc., 3260 Jay St., Santa Clara, CA 95054 and Information Systems Lab, Stanford University. Research support from AFOSR under Contract F49620-89-C-0119.

problem in the next section, the minimax control is proved in section 3 to be the LQR control designed for the worst-case plant from the given ellipsoidal set. By defining an appropriate mapping, which maps an element of the given set to an element of the same set, the existence of this worst-case plant is proved. In section 4, a simple algorithm used to compute the worst-case plant is given. A two-mass-one-spring example is used in section 5 to illustrate the ideas presented. The paper concludes with some remarks in section 6.

2 Problem Formulation

Consider the following family of systems

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0 \tag{1}$$
$$y(t) = c^T x(t), \tag{2}$$

where A, b, and x_0 are fixed and given, and

$$c \in \Theta = \left\{\theta : (\theta - \theta_c)^T R(\theta - \theta_c) \leq 1, \quad R = R^T > 0\right\}. \tag{3}$$

For a given control, $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, and a fixed $c \in \Theta$, the LQR cost is defined to be

$$J(u, c) \triangleq \int_0^\infty \left[ru(t)^2 + y(t)^2\right] dt. \tag{4}$$

We assume that $(A,\ b)$ is controllable (or at least stabilizable) and $(c,\ A)$ is observable (or at least detectable) for all c in Θ. The robust control design problem is to find a control $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ that solves the following minimax problem:

$$\min_u \max_{c \in \Theta} J(u, c). \tag{5}$$

Since no particular form is assumed for the control u, such as linear state-feedback, the minimization in (5) is over all possible u's. Note also that we chose the initial time $t = 0$ for convenience only, the problem can be posed at any initial time $t = t_0$. Therefore, one can design a new controller each time Θ gets updated.

The cost objective in (4) and the ellipsoidal set in (3) lead to another interesting interpretation for the minimax problem in (5) once we rewrite (4) as

$$J(u, c) = \int_0^\infty \left[ru(t)^2 + x^T(t)cc^T x(t)\right] dt. \tag{6}$$

Now, instead of saying that we are designing a controller for a set of uncertain plants described by (1) through (3), we can also say that we are designing a controller for a set of uncertain objective functions. (This interpretation contrasts with the standard LQR design where a controller is obtained for fixed weighting matrices.) Note that $c^T x(t)$ is a dot product, so it depends on the angle between c and $x(t)$. Geometrically, the set Θ sweeps out a "cone" (with a curved base) of possible c's. Thus, we can interpret Θ as a set of "view angles" from which we calculate the cost. The minimax control from (5) is therefore robust to all these "view angles." This interpretation is interesting since in practice we seldom look at performance from just one angle.

3 Minimax Solution

To solve the minimax problem in (5), recall from [6, pages 274-282] that (u^*, c^*) is a saddle point if

$$J(u^*, c) \leq J(u^*, c^*) \leq J(u, c^*) \tag{7}$$

for all $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $c \in \Theta$. In that case, we have

$$(u^*, c^*) = \arg\min_u \max_{c \in \Theta} J(u, c) = \arg\max_{c \in \Theta} \min_u J(u, c). \tag{8}$$

Our goal in this section is to prove that there always exists such (u^*, c^*) for (5).

From LQR control theory, the second inequality in (7) is true if

$$u^* = u_{LQR}(c^*), \tag{9}$$

where $u_{LQR}(c^*)$ denotes the LQR control designed for the plant in (1) with $c = c^*$ in (2). It follows that the first inequality in (7) is also true if

$$c^* = \arg\max_c J\left(u_{LQR}(c^*), c\right). \tag{10}$$

Thus, if c^* exists for (10), the minimax problem in (5) is solved by (9). Note that the existence of c^* is not obvious because c^* must have the property that when $u_{LQR}(c^*)$ is applied to each $c \in \Theta$, the maximum cost occurs at c^*.

We now express the LQR cost in (10) in a more convenient form. Since $(A,\ b)$ is stabilizable and $(c,\ A)$ is detectable for all c in Θ, for each $c \in \Theta$ there is an associated state-feedback control $u_{LQR}(c)$ given by

$$u_{LQR}(c) = -K_c x, \tag{11}$$

where

$$K_c = \frac{1}{r} b^T P_c \tag{12}$$

and P_c satisfies the algebraic Riccati equation

$$A^T P_c + P_c A - \frac{1}{r} P_c b b^T P_c + cc^T = 0. \tag{13}$$

We will use X_c to denote the solution of the associated Lyapunov equation,

$$(A - bK_c)X_c + X_c(A - bK_c)^T + x_0 x_0^T = 0, \tag{14}$$

where

$$X_c = \int_0^\infty e^{(A - bK_c)t} x_0 x_0^T e^{(A - bK_c)^T t}\, dt. \tag{15}$$

The LQR cost in (10) can now be expressed as

$$\begin{aligned} & J(u_{LQR}(c^*), c) \\ &= \int_0^\infty \left[r u_{LQR}(c^*)^2 + y^2 \right] dt \\ &= \int_0^\infty \left[r K_{c^*} x x^T K_{c^*}^T + c^T x x^T c \right] dt \\ &= \int_0^\infty r K_{c^*} e^{(A-bK_{c^*})t} x_0 x_0^T e^{(A-bK_{c^*})^T t} K_{c^*}^T \, dt \\ &\quad + \int_0^\infty c^T e^{(A-bK_{c^*})t} x_0 x_0^T e^{(A-bK_{c^*})^T t} c \, dt \\ &= r K_{c^*} X_{c^*} K_{c^*}^T + c^T X_{c^*} c. \end{aligned} \tag{16}$$

For a given c^*, $K_{c^*} X_{c^*} K_{c^*}^T$ in (16) is fixed. Thus, the maximization in (10) becomes

$$c^* = \arg\max_c c^T X_{c^*} c. \tag{17}$$

Note that the feedback gain K_{c^*} does not depend on the initial condition x_0, but the Lyapunov solution X_{c^*} does. Therefore, the solution c^* is a function of x_0. However, this dependence on x_0 can be removed if we start with the assumption that x_0 is a random vector with known mean m and covariance C and the objective in (4) is an expectation over x_0. In that case, X_{c^*} is the solution of (14) with $x_0 x_0^T$ replaced by $C + mm^T$.

Our ultimate goal is to find c^* in (17), but we must first prove that such c^* always exists. To do that, we define the mapping $f: \bar{c} \in \Theta \rightarrow \hat{c} \in \Theta$,

$$\begin{aligned} f(\bar{c}) &= \hat{c} \\ &\triangleq \arg\max_c c^T X_{\bar{c}} c, \end{aligned} \tag{18}$$

where $X_{\bar{c}}$ satisfies the Lyapunov equation associated with $\bar{c}$ as in (14). It was shown in [15] that the solution of (18) is given by

$$\hat{c} = T \Lambda^{-\frac{1}{2}} \hat{z} + \theta_c \in \Theta_b, \tag{19}$$

where

$$R = T \Lambda T^T \tag{20}$$

$$\hat{z} = (\Omega - \hat{\lambda} I)^{-1} \beta \tag{21}$$

$$\hat{\lambda} = \max \lambda \left(\begin{bmatrix} \Omega & -I \\ -\beta\beta^T & \Omega \end{bmatrix} \right) \tag{22}$$

$$\Omega = R^{-\frac{T}{2}} X_{\bar{c}} R^{-\frac{1}{2}} \tag{23}$$

$$\beta = -R^{-\frac{T}{2}} X_{\bar{c}} \theta_c \tag{24}$$

$$\Theta_b = \left\{ \theta : (\theta - \theta_c)^T R (\theta - \theta_c) = 1 \right\} \tag{25}$$

(Θ_b is the boundary of Θ.) Therefore, the mapping f consists of two parts. First, it takes the given $\bar{c}$ and produces $X_{\bar{c}}$ via equations (12) through (14). Then $\hat{c}$ is given by (19).

To show that c^* exists in (17) is equivalent to showing that a fixed point c^* exists for f, *i.e.*,

$$f(c^*) = c^*. \tag{26}$$

To do that, we need a lemma extracted from [11] and a simple form of Brouwer's Theorem [13, pages 366-367].

Lemma 1 *If (A, b) is stabilizable, then over any region where (c, A) is detectable, the algebraic Riccati equation solution P_c in (13) is continuous in cc^T.*

Proof of Lemma 1 Consider the matrix-valued functional

$$g\left(P, cc^T\right) = A^T P + PA - \frac{1}{r} P b b^T P + cc^T. \tag{27}$$

For any c, P_c satisfies (13), so $g\left(P_c, cc^T\right) = 0$. As a quadratic function in P and a linear function in cc^T, the functional g is infinitely differentiable, and its derivative with respect to P at the point $\left(P_c, cc^T\right)$ is the linear operator given for any matrix Z by

$$Dg_P(Z) = (A - bK_c)^T Z + Z(A - bK_c). \tag{28}$$

Since K_c is stabilizing, the operator Dg_P is nonsingular by Lyapunov's equation. Therefore, from the implicit function theorem (see, *e.g.*, [21, pages 375-380]), there exists an infinitely differentiable matrix-valued function Ψ such that

$$P_c = \Psi(cc^T). \tag{29}$$

Thus, P_c is continuous in cc^T. □

Theorem 2 (Brouwer's Theorem) *Let C be a compact, convex subset of $\mathbb{R}^n$. Then any continuous function $f: C \rightarrow C$ has at least one point c^* such that $f(c^*) = c^*$.*

The existence of c^* in (17) can now be guaranteed by the following theorem.

Theorem 3 (Fixed Point) *The mapping f defined in (18) is continuous in $\bar{c}$ and it has a fixed point.*

Proof of Theorem 3 First, we need to show that the mapping from $\bar{c}$ to $X_{\bar{c}}$ is continuous.

1. Let $c = \bar{c}$ in (12) through (14). By Lemma 1, $P_{\bar{c}}$ of (13) is continuous in $\bar{c}\bar{c}^T$. Since each element of $\bar{c}\bar{c}^T$ is simply a product of elements from $\bar{c}$, $\bar{c}\bar{c}^T$ is continuous in $\bar{c}$. By the continuity of composite functions, $P_{\bar{c}}$ is continuous in $\bar{c}$.

2. $K_{\bar{c}}$ of (12) is continuous in $P_{\bar{c}}$, thus it is continuous in $\bar{c}$.

3. By the implicit function theorem (similar to the proof of Lemma 1), $X_{\bar{c}}$ is continuous in $K_{\bar{c}}$. By the continuity of composite functions, $X_{\bar{c}}$ is continuous in $\bar{c}$.

Second, we need to show that the mapping from X_ε to $\bar{c}$ is also continuous.

1. Both Ω and β in (23) and (24) are continuous in X_ε. Since each eigenvalue of a matrix is continuous in the elements of the matrix (see, *e.g.*, [10, pages 191-192]), $\hat{\lambda}$ in (22) is continuous in X_ε. Thus, by the continuity of composite functions, $\bar{\lambda}$ is continuous in $\bar{c}$.

2. Each element of $(\Omega - \hat{\lambda}I)^{-1}$ is given by its cofactor divided by $\det(\Omega - \hat{\lambda}I)$. The cofactors and $\det(\Omega - \hat{\lambda}I)$ are sums of products of elements of $\Omega - \hat{\lambda}I$. Thus, $(\Omega - \hat{\lambda}I)^{-1}$ is continuous in $\bar{c}$, which implies $\hat{z}$ in (21) is continuous in $\bar{c}$ also. (Exception is when $\Omega - \hat{\lambda}I$ is singular, which is treated in [15]. However, continuity is not affected.)

3. $\hat{c}$ in (19) is continuous in $\bar{c}$.

Therefore, the mapping f from $\bar{c}$ to $\hat{c}$ is continuous, and by Brouwer's Theorem it has at least one fixed point. □

The existence of a saddle-point solution for the minimax problem in (5) is stated in the following theorem.

Theorem 4 (Existence) *There exists at least one (u^*, c^*) such that (7) is true and the minimax problem in (5) has a saddle-point solution. If there are more than one (u, c) which satisfy (7), then their associated LQR costs must be equal and any one of the solutions is equally valid.*

Proof of Theorem 4 From Theorem 3, we know that (10) has at least one fixed point. Therefore, (7) has at least one saddle-point solution. To show that two fixed points of (10) must have the same LQR cost, assume that there exist (u_1, c_1) and (u_2, c_2) such that

$$J(u_1, c) \le J(u_1, c_1) \le J(u, c_1), \quad \forall\ u,\ c \tag{30}$$

and

$$J(u_2, c) \le J(u_2, c_2) \le J(u, c_2), \quad \forall\ u,\ c. \tag{31}$$

Then let $c = c_2$ and $u = u_2$ in (30), we get

$$J(u_1, c_2) \le J(u_1, c_1) \le J(u_2, c_1) \le J(u_2, c_2) \tag{32}$$

or

$$J(u_1, c_1) \le J(u_2, c_2). \tag{33}$$

Similarly, let $c = c_1$ and $u = u_1$ in (31), we get

$$J(u_2, c_1) \le J(u_2, c_2) \le J(u_1, c_2) \le J(u_1, c_1) \tag{34}$$

or

$$J(u_2, c_2) \le J(u_1, c_1). \tag{35}$$

Therefore, (33) and (35) imply

$$J(u_1, c_1) = J(u_2, c_2). \tag{36}$$

□

This section can be summarized as follows: a fixed-point solution c^* exists for (10) and the solution to the minimax problem in (5) is given by (9). We now turn to the computation of c^*.

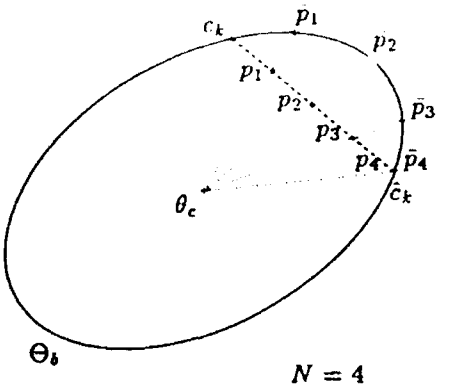

Figure 1: Candidate points used in calculating $\bar{c}_{k+1}$ and $\hat{c}_{k+1}$.

4 Fixed-Point Computation

Before describing our simple heuristic algorithm, we should point out that there exist many algorithms to compute Brouwer fixed points (see *e.g.*, [1] and [24].) Although these algorithms can guarantee that the fixed points will be found, they are known to have combinatorial complexity. In comparison, we have no guarantee that our algorithm will converge, but in many cases that we have tried, it usually converges in less than 10 iterations.

The goal of the iterative algorithm below is to find $\bar{c}_k$ such that the distance between $\bar{c}_k$ and $\hat{c}_k = f(\bar{c}_k)$, as defined in (18), is small, *i.e.*, a fixed point. Given $\bar{c}_k$ and $\hat{c}_k$ at the kth iteration, steps 6 through 8 below are designed to find $\bar{c}_{k+1}$ and $\hat{c}_{k+1}$. The algorithm accomplishes this by doing a local minimization over a set of candidate points, $\{\bar{p}_i,\ i = 1, \ldots, N\}$. Let $\{p_i,\ i = 1, \ldots, N\}$ be $N - 1$ equally-spaced points between $\bar{c}_k$ and $\hat{c}_k$, with $p_N = \hat{c}_k$ (see Figure 1). Vectors are then drawn from θ_c to each p_i, until they intersect Θ_b at points $\{\bar{p}_i,\ i = 1, \ldots, N\}$, where

$$\bar{p}_i = \gamma w + \theta_c \tag{37}$$

$$\gamma = \sqrt{\frac{1}{w^T R w}} \tag{38}$$

$$w = \frac{p_i - \theta_c}{\|p_i - \theta_c\|_2}. \tag{39}$$

Next, we compute $\hat{p}_i = f(\bar{p}_i)$ in step 7. After comparing

the distances $\|\bar{p}_i - \hat{p}_i\|_2$, the $\bar{p}_j$ and $\hat{p}_j$ with the minimum distance become $\bar{c}_{k+1}$ and $\hat{c}_{k+1}$, respectively.

A Heuristic Algorithm

1. Define the mapping f from $\bar{c}$ to $\hat{c}$: compute X_c in (18) using (13), (12), and (14) then compute $\hat{c}$ using (19);
2. $k \leftarrow 0$;
3. Let $\bar{c}_1$ be a random point on Θ_b;
4. Compute $\hat{c}_1 = f(\bar{c}_1)$;
5. $k \leftarrow k + 1$;
6. Compute $\{\bar{p}_i,\ i = 1, \ldots, N\}$ on Θ_b using (37);
7. Compute $\hat{p}_i = f(\bar{p}_i)$ for $i = 1, \ldots, N$;
8. Compute
$$j = \arg\min_i \|\hat{p}_i - \bar{p}_i\|_2 \tag{40}$$
then
$$\bar{c}_{k+1} = \bar{p}_j \tag{41}$$
$$\hat{c}_{k+1} = \hat{p}_j; \tag{42}$$
9. If $\|\hat{c}_{k+1} - \bar{c}_{k+1}\|_2 > \epsilon$, go to step 5.

Note that there is no guarantee that $\|\hat{c}_k - \bar{c}_k\|_2 < \|\hat{c}_{k+1} - \bar{c}_{k+1}\|_2$, so we don't have a convergence proof for this algorithm. However, with $\epsilon = 0.001$, this algorithm usually converges in less than 10 iterations.

5 Example

We will use the two-mass-one-spring system described in [4] in our example. This system, shown in Figure 2, can be represented in state-space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u \tag{43}$$

$$y = c^T x \tag{44}$$

where x_1 and x_2 are the positions of masses 1 and 2, and x_3 and x_4 are the velocities of masses 1 and 2, respectively. We use masses $m_1 = m_2 = 1$ kg and spring coefficient $k = 1$ N/m for this system.

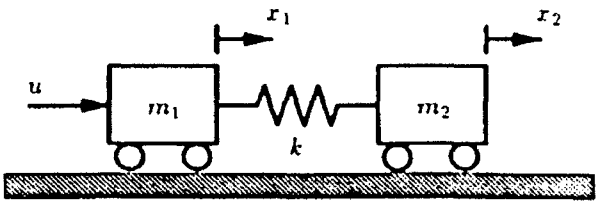

Figure 2: Two-mass-one-spring system.

The initial condition is $x_0 = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$, which means the masses are displaced toward each other. For the ellipsoidal set in (3), we use $\theta_c = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T$ and $R = I$. Thus, the output y is nominally the sum of the position and velocity of the second mass, but c can still be anywhere within the unit ball. We choose $r = 1$ in the objective and $N = 4$ in the fixed-point algorithm. For the stopping criterion, $\epsilon = 0.001$ is used. The algorithm converges in 5 iterations.

Table 1 shows the cost matrix for this example, where c_{LQR} is the element in Θ which maximizes the cost for $u = u_{LQR}(\theta_c)$. As expected, the control $u = u_{LQR}(\theta_c)$ applied to $c = \theta_c$ gives the lowest cost for this control, 5.6, but its cost can be quite high at other c's such as c_{LQR} and c^*. In comparison, the control $u = u_{LQR}(c^*)$ applied to $c = \theta_c$ gives a slightly higher cost (but this may not be the lowest cost for this control as it is likely that another c achieves the minimum) while keeping the maximum cost to 13.4, as compared to a maximum of 17.1 for $u = u_{LQR}(\theta_c)$. Therefore, this example illustrates that by giving up some performance at the nominal plant θ_c, we gain some performance back for other plants in the set.

	$c = \theta_c$	$c = c_{LQR}$	$c = c^*$
$u = u_{LQR}(\theta_c)$	5.6	17.1	16.9
$u = u_{LQR}(c^*)$	7.3	13.3	13.4

Table 1: Cost matrix for different u's and c's.

6 Conclusion

We presented a controller design method for continuous-time plants whose uncertain parameters in the output matrix are known to lie in an ellipsoidal set. This design problem is posed as a minimax problem, where the control and plant uncertainty can be viewed as strategies employed by opposing players in a game, in which the control is chosen to minimize the LQR cost and the plant uncertainty is chosen to maximize it. Without restricting the form of this minimax control, we proved that it is the LQR control for one of the plants in the ellipsoidal set.

the worst-case plant. We then proved that this worst-case plant always exists as a fixed point for a certain mapping. A simple heuristic algorithm for computing this fixed point was also given.

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A Robust Control Design for FIR Plants with Parameter Set Uncertainty

MING K. LAU*† STEPHEN BOYD* ROBERT L. KOSUT‡ GENE F. FRANKLIN*

Abstract This paper proposes a method of computing the finite-horizon control inputs for FIR plants whose parameters are only known to lie in a set. The parameter set is assumed to be described by an ellipsoidal bound, which could be provided by some identification scheme with a parameter set estimator. The finite-horizon control obtained minimizes the maximum LQR cost from all plants with parameters in the given set. The computation of this robust control is shown to be a convex optimization problem, thus global minimization is guaranteed and many efficient methods are available to compute the minimizing control. In addition, the method can also be used to compute the control for the dual problem in which the plant parameters are known but the initial states of the plant are assumed to lie in a set.

1 Introduction

A problem of great interest in control theory is the design of a controller which can guarantee some level of performance in the presence of plant parameter uncertainty. Kharitonov's theorem provides a necessary and sufficient analysis test for determining the robust stability of polynomials with perturbed coefficients, however, there are few results that exploit Kharitonov's theorem for synthesizing robust controllers, *e.g.*, [4] and [10]. Another approach to this problem is to define a set of nominal values of the uncertain parameters and consider deviations from these nominal values. A comprehensive survey of the different parameter space methods, as opposed to frequency domain methods, can be found in [13].

Motivated by recent work from [11], [12], and [1], where the identified plant parameters are described by ellipsoidal sets, we pose the following problem: given that the plant parameters are known to lie in an ellipsoid, find the finite-horizon control which minimizes the maximum LQR cost from all plants with parameters in the given set. At time k, this minimization produces the control vector $[u(k) \;\; u(k+1) \;\; \cdots \;\; u(k+N)]$, but only $u(k)$ is applied. At time $k+1$, a new minimization problem is solved. This approach of control application is the same as the generalized predictive control described in [5] and [2].

In this paper, we choose to work with finite impulse response (FIR) models for the plant with the assumption that they are accurate models provided they of sufficient lengths. (In doing so, we have also assumed that the plant is stable.) Our goals are to show that the above minimization problem is a convex optimization problem and to design an algorithm to compute the minimizing control. In order to solve the minimization problem, a constrained maximization problem must also be solved. The procedures of which are given in the Appendix. We will also show that the same algorithm can be used to compute the control for the dual problem in which the plant parameters are known but the initial states of the plant are assumed to lie in a set. The paper is organized as follows, after stating the problem in the next section, we will show convexity in section 3 and outline the algorithm. The dual problem of uncertain initial states is considered in section 4. A numerical example is given in section 5. Some concluding remarks are given in section 6.

2 Problem Statement

We shall consider a discrete FIR plant

$$\begin{aligned} y(k) &= b_1 u(k-1) + \cdots + b_m u(k-m) \\ &= \theta^T \phi(k) \end{aligned} \qquad (1)$$

where $y(k)$ and $u(k)$ are the output and control of the plant at time k, respectively, and

$$\begin{aligned} \theta &= \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}^T \\ \phi(k) &= \begin{bmatrix} u(k-1) & u(k-2) & \cdots & u(k-m) \end{bmatrix}^T \end{aligned}$$

The parameter vector of the plant, θ, is assumed to be in a set,

$$\theta \in \Theta \triangleq \{\theta : (\theta - \theta_c)^T \Gamma (\theta - \theta_c) \leq 1\} \qquad (2)$$

where $\Gamma = \Gamma^T > 0$. Note that Θ describes an ellipsoid in the parameter space with its center at θ_c. The matrix Γ gives the size and orientation of the ellipsoid, *i.e.*, the

*Information Systems Laboratory, Stanford University, Stanford, CA 94305.

†Supported by Sandia National Labs' Doctoral Study Program.

‡Integrated Systems Inc., 2500 Mission College Blvd., Santa Clara, CA 95054 and Information Systems Lab, Stanford University.

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square roots of the reciprocals of the eigenvalues of Γ are the lengths of the semi-axes of the ellipsoid and the eigenvectors of Γ are the directions of the semi-axes.

The plant in (1) can also be represented in state space format,

$$x(k+1) = Ax(k) + bu(k) \quad (3)$$
$$y(k) = cx(k) \quad (4)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and

$$c = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix} = \theta^T$$

Thus, the states of the FIR plant are

$$x(k) = \begin{bmatrix} u(k-1) & u(k-2) & \cdots & u(k-m) \end{bmatrix}^T = \phi(k)$$

Due to past disturbances, the states at some time k_0 are displaced to $\phi(k_0) = \phi_0 \neq 0$, so $y(k_0) \neq 0$. Without loss of generality, we let $k_0 = 0$. We now define the control and output vectors

$$u \triangleq \begin{bmatrix} u(0) & u(1) & u(2) & \cdots & u(N) \end{bmatrix}^T$$
$$y \triangleq \begin{bmatrix} y(0) & y(1) & y(2) & \cdots & y(N) \end{bmatrix}^T$$

and the quadratic cost function

$$J_o \triangleq \rho u^T u + y^T y \quad (5)$$

where ρ is a weight to trade control effort for regulation. The problem is to find a control which minimizes the cost function for the worst possible plant in Θ, *i.e.*,

$$u^* = \arg\min_u \left(\max_{\theta \in \Theta} J_o \right) \quad (6)$$

Thus, u^* is designed to be robust with respect to the parameter set uncertainty given in (2). Note that if there were no parameter uncertainty in the plant, $\theta = \theta_c$, then (6) becomes

$$u_{LQR} = \arg\min_u J_o \quad (7)$$

which is the standard finite-horizon linear quadratic regulator problem. The optimal control in (7) requires the solution of the discrete Riccati equation, which can be found in texts such as [7, 2].

3 Robust Control Design

We will solve the minimax problem of (6) by showing that it is a convex optimization problem. Note that since $u^T u$ is not a function of θ, we have

$$u^* = \arg\min_u \left[J_1(u) + J_2(u) \right]$$

where

$$J_1(u) = \rho u^T u \quad (8)$$
$$J_2(u) = \max_{\theta \in \Theta} y^T y \quad (9)$$

We can express y as

$$y = U\theta$$

where

$$U = \begin{bmatrix} u(-1) & u(-2) & \cdots & u(-m) \\ u(0) & u(-1) & \cdots & u(-m+1) \\ u(1) & \ddots & \ddots & \vdots \\ \vdots & & & \vdots \\ u(N-1) & u(N-2) & \cdots & u(N-m) \end{bmatrix}$$

We now state and prove the following corollary, which states that the maximizer of (9) always lies on the boundary of Θ.

Corollary 1 *Let $\|\cdot\|_2$ denote the Euclidean norm,* i.e.

$$\|x\|_2^2 \triangleq x^T x$$

For a fixed matrix U,

$$f(\theta) = \|U\theta\|_2^2$$

is convex in θ and

$$\max_{\theta \in \Theta} \|U\theta\|_2^2 = \max_{\theta \in \Theta_b} \|U\theta\|_2^2 \quad (10)$$

where

$$\Theta_b = \{\theta : (\theta - \theta_c)^T \Gamma (\theta - \theta_c) = 1\} \quad (11)$$

Proof of Corollary 1 Let $\alpha \in [0, 1]$, then

$$\begin{aligned} & f(\alpha\theta_1 + (1-\alpha)\theta_2) - \alpha f(\theta_1) - (1-\alpha) f(\theta_2) \\ = \; & \|U(\alpha\theta_1 + (1-\alpha)\theta_2)\|_2^2 - \alpha \|U\theta_1\|_2^2 - (1-\alpha) \|U\theta_2\|_2^2 \\ = \; & -\alpha(1-\alpha) \|U(\theta_1 - \theta_2)\|_2^2 \\ \leq \; & 0 \end{aligned}$$

Thus, $f(\theta)$ is convex in θ. Now let $\theta_1, \theta_2 \in \Theta_b$, then

$$f(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha f(\theta_1) + (1-\alpha) f(\theta_2)$$

Since the graph of $f(\theta)$ along the line segment joining any θ_1 and θ_2 lies on or below the line segment with its ends at $f(\theta_1)$ and $f(\theta_2)$, (10) follows. (A different proof

of the maximum occurring on the boundary can be found in [14].) □

Thus, the maximizer of (9) is given by

$$\theta^* = \arg \max_{\theta \in \Theta_b} \|U\theta\|_2^2 \tag{12}$$

Theorem 1 *The functional*

$$J(u) = J_1(u) + J_2(u) \tag{13}$$

is convex in u.

Proof of Theorem 1 We express y as

$$y = B_1(\theta)u + B_2(\theta)\phi_o \tag{14}$$

where

$$B_1(\theta) = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ b_1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ b_2 & b_1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ b_m & b_{m-1} & \cdots & b_1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & b_m & b_{m-1} & \cdots & b_1 & 0 \end{bmatrix}$$

$$B_2(\theta) = \begin{bmatrix} b_1 & b_2 & \cdots & b_{m-1} & b_m \\ b_2 & b_3 & \cdots & b_m & 0 \\ b_3 & \cdots & b_m & 0 & 0 \\ \vdots & b_m & & & \vdots \\ b_m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

and

$$\phi_o = \begin{bmatrix} u(-1) & u(-2) & \cdots & u(-m) \end{bmatrix}^T$$

Then

$$y^T y = \phi_o^T B_2^T B_2 \phi_o + 2\phi_o^T B_2^T B_1 u + u^T B_1^T B_1 u \tag{15}$$

The first term on the right-hand side of (15) is constant in u, the second term is linear in u, and the third term $u^T B_1^T B_1 u = \|B_1 u\|_2^2$ is convex in u by Corollary 1. Thus, $y^T y$ is convex in u for each $\theta \in \Theta$. Since the maximum of a set of convex functionals is also convex [3, page 131], $J_2(u)$ is convex. By Corollary 1, $J_1(u) = \rho \|u\|_2^2$ is convex also. Since the sum of convex functionals is convex [3, page 131], $J(u)$ is convex in u. □

With Theorem 1, we are guaranteed that there is a global minimum solution for u^* and many efficient methods are available to compute it. However, we want to point out that although $J_2(u)$ is convex in u, it is not differentiable for all u. We will illustrate this point with the following simple example. Consider the case where $m = 2$ and $N = 1$, so $\theta = [b_1 \;\; b_2]^T$ and $u = [u(0) \;\; u(1)]^T$. Since $y(1)$ does not depend on $u(1)$, we have $u(1) = 0$ and can consider $u = u(0)$. Let Θ be the set of points which lie on the line segment from $\theta_1 = [0.5 \;\; -1]^T$ to $\theta_2 = [1 \;\; 1]^T$, and $\phi_o = [-1 \;\; 1]^T$. As shown in Corollary 1, for a given u, the maximum of $y^T y$ must be at either endpoints of Θ,

$$J_2(u) = \max \left(y^T y|_{\theta=\theta_1}, y^T y|_{\theta=\theta_2} \right)$$

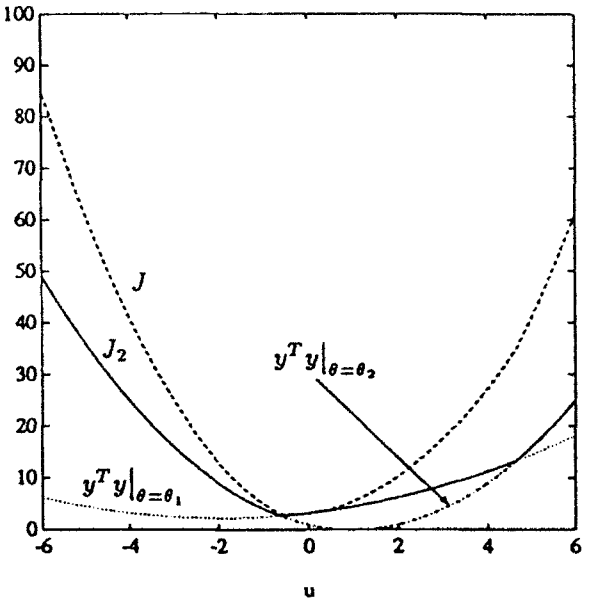

Figure 1: J and J_2 as functions of u.

Figure 1 shows that for this example, there are two points where $J_2(u)$ is not differentiable. Also shown in Figure 1 is $J(u)$ with its minimum at $u^* = -0.4$.

Since $J_2(u)$ is not differentiable for all u, we choose not to use the usual descent methods to find u^*. Instead, we will show that we can easily compute a subgradient of $J(u)$ and apply the ellipsoid algorithm described in [3, pages 324–332].

We first give the definition of a subgradient. If $J : \mathbb{R}^{N+1} \to \mathbb{R}$ is convex, but not necessarily differentiable, then $g \in \mathbb{R}^{N+1}$ is a subgradient of J at u_o if

$$J(u) \geq J(u_o) + g^T(u - u_o) \quad \text{for all } u$$

The set of all subgradients of J at u_o is denoted by $\partial J(u_o)$, the subdifferential of J at u_o. The following two facts from [3, page 300] will be used.

1. Since $J_1(u)$ and $J_2(u)$ are convex, any subgradient of the form $g = g_1 + g_2$ is in $\partial J(u)$, where $g_1 \in \partial J_1(u)$ and $g_2 \in \partial J_2(u)$.

2. Let $y^T y$ from (15) evaluated at θ^* from (12) be de-

noted by

$$J_2(u,\theta^*) = \phi_o^T B_2^T(\theta^*)B_2(\theta^*)\phi_o + 2\phi_o^T B_2^T(\theta^*)B_1(\theta^*)u + u^T B_1^T(\theta^*)B_1(\theta^*)u \quad (16)$$

Since $y^T y$ is convex in u for each $\theta \in \Theta_b$, $g_2 \in \partial J_2(u,\theta^*)$ implies $g_2 \in \partial J_2(u)$. In the event that there are more than one maximum, we only need to pick one.

Thus, from (8) and (16) the subgradient of J at u is given by

$$g = 2\rho u + \left[2B_1^T(\theta^*)B_2(\theta^*)\phi_o + 2B_1^T(\theta^*)B_1(\theta^*)u\right] \quad (17)$$

The computation of θ^* is not difficult, but the derivation is rather long. To avoid breaking the flow of this section, the method of finding θ^* is given in the Appendix. The ellipsoid algorithm for computing $u^* \in \mathbb{R}^K$ is as follows:

1. Select any u_1 and E_1 such that u^* is in the initial ellipsoid,
$$u^* \in \left\{u : (u-u_1)^T E_1^{-1}(u-u_1)\right\}$$
2. $k \leftarrow 0$;
3. $k \leftarrow k+1$;
4. Compute any $g_k \in \partial J(u_k)$:
   (a) Compute z^* from Theorem 2;
   (b) Compute θ^* from (31);
   (c) Compute g_k from (17);
5. Compute new ellipsoid:
$$\begin{aligned} \tilde{g} &\leftarrow \frac{g_k}{\sqrt{g_k^T E_k g_k}} \\ u_{k+1} &\leftarrow u_k - \frac{E_k\tilde{g}}{K+1} \\ E_{k+1} &\leftarrow \frac{K^2}{K^2-1}\left(E_k - \frac{2}{K+1}E_k\tilde{g}\tilde{g}^T E_k\right) \end{aligned}$$
6. If $\sqrt{g_k^T E_k g_k} > \epsilon$, go to step 3.

The stopping criterion in step 6 guarantees that on exit, $J(u_k)$ is within ϵ of $J(u^*)$.

4 Uncertain Initial States

In this section, we will consider the dual problem in which the parameter vector θ of the plant is known, but the initial states of the plant ϕ_o is assumed to be in a set similar to (2),

$$\phi_o \in \Phi \triangleq \{\phi_o : (\phi_o - \phi_c)^T \Gamma_\phi (\phi_o - \phi_c) \le 1\} \quad (18)$$

The problem posed in (6) now becomes

$$\begin{aligned} u^* &= \arg\min \max_{\phi_o \in \Phi} J_o \quad (19) \\ &= \arg\min \left[\rho u^T u + \max_{\phi_o \in \Phi} y^T y\right] \end{aligned}$$

Note that $y^T y$ from (15) is convex in ϕ_o for a given u. This means that the maximum of $y^T y$ lies on the boundary of Φ, Φ_b. Furthermore, using the same arguments from the proof of Theorem 1,

$$J_\phi(u) = \rho u^T u + \max_{\phi_o \in \Phi_b} y^T y$$

can be shown to be convex in u. Therefore, all we need to show is that we can compute a subgradient of $J_\phi(u)$,

$$g_\phi = 2\rho u + 2B_1^T B_1 u + 2B_1^T B_2 \phi_o^* \quad (20)$$

where

$$\phi_o^* = \arg\max_{\phi_o \in \Phi_b} y^T y$$

From (14), we have

$$\phi_o^* = \arg\max_{\phi_o \in \Phi_b} \|B_2\phi_o + B_1 u\|_2$$

This is similar to the form of (12) except that we have the extra term $B_1 u$. Thus, if we solve for θ^* with

$$q = -(B_2\phi_c + B_1 u)$$

in (29) and replace Γ and θ_c of (2) with Γ_ϕ and ϕ_c from (18), we have

$$\phi_o^* = \theta^*$$

Therefore, u^* in (19) can be computed by the same ellipsoid algorithm given in Section 4, where the subgradient is now computed using (20).

5 Numerical Example

For our example, we use a 10-tap FIR plant, *i.e.*, $m = 10$. The control vector u has $N = 10$, so if $u \equiv 0$, the output will be zero after 10 delays, $y(10) = 0$. The parameter ellipsoid Θ in (2) is a 10-dimensional ball with a radius of 5 and center at θ_c. θ_c, plotted in Figure 2 with the '+' symbol, is the first ten terms of the impulse response from the transfer function

$$\frac{10z(z+0.7\cos(\pi/4))}{z^2 - 2(0.7)\cos(\pi/4)z + 0.7^2}$$

The initial state of the plant,

$$x(0) = [u(-1) \;\; u(-2) \;\; \cdots \;\; u(-10)]^T$$

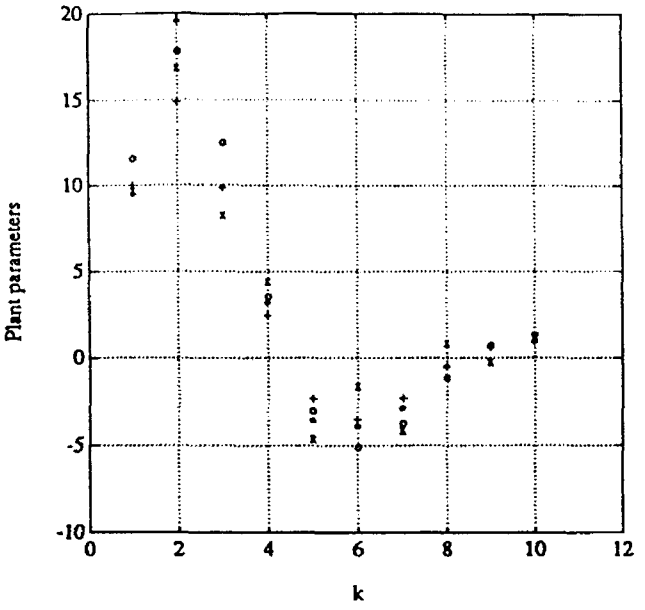

Figure 2: Plant parameters: o – θ_1, x – θ_2, * – θ_3, and + – θ_c.

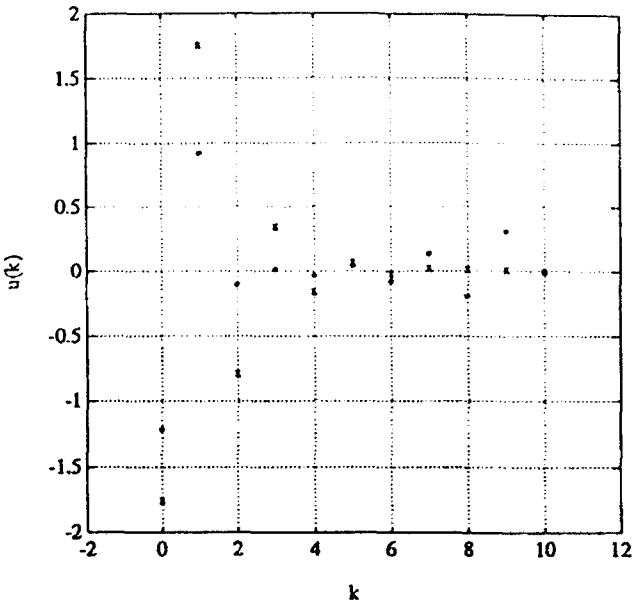

Figure 3: Controls: x – u_{LQR} and * – u^*.

is scaled such that $\|x(0)\|_2 = 1$.

Using $\rho = 1$, we will compare the cost J in (13) associated with three controls, $u_1 = 0$, $u_2 = u_{LQR}$, and $u_3 = u^*$, where u_{LQR} is given by (7) with $\theta = \theta_c$. The controls u_{LQR} and u^* are plotted in Figure (3), where $\|u_{LQR}\|_2 = 2.63$ and $\|u^*\|_2 = 1.58$. We now define three plants from Θ,

$$\theta_i \triangleq \arg\max_{\theta \in \Theta} (J_o|_{u=u_i}) \quad i = 1, 2, 3$$

They are the worst-case plants for their associated controls and are plotted in Figure 2. Table 1 shows the cost matrix, C, for the different plants and controls. We make the following observations from C:

1. For $i = 1,\ 2,\ 3$, $C(i,i)$ is the largest in each row, as the θ_i's are chosen that way.
2. u_{LQR} has the lowest cost for θ_c, 403, but only 8% lower than u^*.
3. u^* has the lowest maximum cost, 697, 48% lower than the maximum cost from u_{LQR} and 87% lower than that from $u = 0$. Thus, the robust design performed as expected.

	θ_1	θ_2	θ_3	θ_c
$u_1 = 0$	1306	902	1136	837
$u_2 = u_{LQR}$	587	1031	785	403
$u_3 = u^*$	655	632	697	437

Table 1: Cost matrix for different u's and θ's.

6 Concluding Remarks

We have shown in this paper that given that the FIR plant parameters are known to lie in an ellipsoid, finding the finite-horizon control to minimize the maximum LQR cost from all plants with parameters in the given set is a convex optimization problem. An algorithm is given to compute this minimizing control. Although the algorithm can also compute the minimizing control when the plant parameters are known but the initial states of the plant are in an ellipsoid, it would be desirable to minimize the maximum over both parameter and initial state uncertainties simultaneously. Furthermore, we would like to extend our method to the infinite-horizon case for infinite impulse response (IIR) plants. These are areas of our current research.

7 Appendix

Given the following matrices,

$$U \in \mathbb{R}^{(N+1)\times m} \tag{21}$$

$$\Gamma \in \mathbb{R}^{m\times m},\ \Gamma = \Gamma^T > 0 \tag{22}$$

$$\theta, \theta_c \in \mathbb{R}^m \tag{23}$$

we want to find the maximizer θ^* in (12). This is similar to the least squares problem with quadratic and linear constraints, which was investigated in [8] and [9]. However, we are seeking a maximizer as compared to a minimizer.

Since Γ is symmetric, we can diagonalize it by a unitary matrix,

$$\Gamma = T\Lambda T^T$$

where Λ is diagonal with eigenvalues of Γ and the columns of T are eigenvectors of Γ. We now transform Θ_b in (11) to the unit ball,

$$B = \left\{ z : z^T z = 1 \right\} \tag{24}$$

where

$$z = \Lambda^{\frac{1}{2}} T^T (\theta - \theta_c) \tag{25}$$

Substituting

$$\theta = T\Lambda^{-\frac{1}{2}} z + \theta_c \tag{26}$$

into (12), we have

$$x^* = \arg \max_{x^T x = 1} \|Dz - q\|_2^2 \tag{27}$$

where

$$D = UT\Lambda^{-\frac{1}{2}} \tag{28}$$

$$q = -U\theta_c \tag{29}$$

Define

$$\Omega \triangleq D^T D$$

$$\beta \triangleq D^T q$$

then

$$z^* = \arg \max_{z^T z = 1} z^T \Omega z - 2\beta^T z \tag{30}$$

Substituting z^* into (26), θ^* in (12) is given by

$$\theta^* = T\Lambda^{-\frac{1}{2}} z^* + \theta_c \tag{31}$$

To find z^* in (30), we introduce the Lagrange multiplier λ and adjoin the constraint, $z^T z = 1$,

$$L = z^T \Omega z - 2\beta^T z + \lambda \left(1 - z^T z\right)$$

Necessary conditions for the stationary points are

$$\frac{\partial L}{\partial z} = 2\Omega z - 2\beta - 2\lambda z = 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - z^T z = 0$$

or

$$\Omega z = \lambda z + \beta \tag{32}$$

$$z^T z = 1 \tag{33}$$

The problem of finding all the stationary points of such a second-degree polynomial on the unit sphere was first investigated in [6], but the computation of the solution was not considered there. A proof similar to the one given in [8], however, can be used to show the following:

Corollary 2 *If (z_1, λ_1) and (z_2, λ_2) satisfy (32) and (33) and $\lambda_1 > \lambda_2$, then*

$$z_1^T \Omega z_1 - 2\beta^T z_1 > z_2^T \Omega z_2 - 2\beta^T z_2 \tag{34}$$

Thus, in place of the maximization problem in (30), we need to solve the Lagrange equations (32) and (33) with

$$\lambda = maximum \tag{35}$$

In [9], it was shown that (32) and (33) can be transformed to a quadratic eigenvalue problem,

$$(\Omega - \lambda I)^2 \eta = \beta\beta^T \eta$$

Furthermore, the quadratic eigenvalue problem can be reduced to an ordinary eigenvalue problem by finding the eigenvalues of

$$M = \begin{bmatrix} \Omega & -I \\ -\beta\beta^T & \Omega \end{bmatrix}$$

The solution of (30) is summarized in the following theorem:

Theorem 2 *Let λ^* be the largest eigenvalue of M, then there are two possible cases for the maximizer of (30):*

1. *If λ^* is not an eigenvalue of Ω, then $z^* = (\Omega - \lambda^* I)^{-1}\beta$.*
2. *If λ^* is an eigenvalue of Ω, then let $\nu = (\Omega - \lambda^* I)^{\dagger}\beta$, where $\dagger$ denotes the pseudoinverse, and*
   (a) *If $z = \nu$ satisfies (32) and (33), then $z^* = \nu$.*
   (b) *If $z = \nu$ satisfies (32) and $\nu^T \nu < 1$, then $z^* = \nu + \zeta$ is one of many solutions, where ζ is an eigenvector to the eigenvalue λ^* of Ω with $\zeta^T \zeta = 1 - \nu^T \nu$.*

Proof of Theorem 2 In [9], the minimization of (30) was analyzed. Due to Corollary 2, we can apply all the results from [9] by replacing the smallest eigenvalue of M with the largest eigenvalue. □

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Statistical Analysis of Least-Squares Identification for Robust Control Design: Output Error Case With Affine Parametrization

Robert L. Kosut*
Integrated Systems, Inc.
3260 Jay St.
Santa Clara, CA 95054, U.S.A.

Brian D. O. Anderson†
Dept. Of Systems Engineering
The Australian National University
Canberra, ACT 2601, Australia

Abstract Precise, finite-data statistical propereties are determined using a least-squares estimator based on an output error model with an affine parameter representation where the true system is of output error form, *but is not in the model set.* The purpose of the analysis is to show the effect of unmodeled dynamics on the resulting closed-loop system designed on the basis of the estimated transfer function. This simple problem set-up is prototypical of the interplay between system identification and robust control design.

Introduction

The problem addressed is the following: given a finite collection of sensed sampled input/output data from an unknown system, what level of confidence can be assigned to a feedback controller design or modification.

To make the problem both representative and analytically tractable, the following *a priori qualitative* data is assumed:

(a1) The system which is generating the data is a discrete linear-time-invariant system in *output error* form, *i.e.*,

$$y_t = (Gu)_t + e_t \tag{1}$$

where t is the sampling time, u and y are the sensed input and output sequences, respectively, and e is an unpredictable output disturbance. The operator G is linear-time-invariant with unknown transfer function $G(z)$ and corresponding impulse response sequence g. Thus,

$$(Gu)_t = \sum_{k=1}^{\infty} g_k u_{t-k} \tag{2}$$

(a2) $G(z)$ is stable, *i.e.*, all the poles of $G(z)$ are strictly inside the unit circle. Hence, there exist positive constants $M \geq 1$ and $\rho < 1$ such that

$$|g_k| \leq M\rho^{k-1}, \ \forall k \geq 1 \tag{3}$$

(a3) The unpredictable sequence e is zero-mean gaussian i.i.d. with unknown variance λ_e.

(a4) The input sequence u is deterministic, hence, independent of e.

It is important to emphasize that none of the parameters that appear in the above assumptions are assumed to be known; they are only known to exist. Hence, there is no *quantitative a priori* knowledge about M, ρ, or λ_e.

The above qualitative assumptions do, however, impose varying degrees of restrictiveness. Assumption (a1) imposes an LTI structure, which by itself is not necessarily restrictive, however, the output error form is very specific. This latter restriction, together with the gaussian assumption (a3) makes the statistical analysis easier without resorting to a central limit theorem or a law of large numbers. Assumption (a4) implies that the system is operating in open-loop, for otherwise u would have a component which is correlated with e.

For control design it is desireable to obtain an estimate of $G(z)$. It is standard practice to form a parametric model $G(z,\theta)$ and estimate the free parameter θ. Although many parametric forms are possible, *e.g.*, [4], for ease of analysis we choose the following *affine* FIR paramtrization:

$$G(z,\theta) = \sum_{k=1}^{n} \theta_k z^{-k} \tag{4}$$

Thus, the problem is to estimate the first n impulse response coefficients $\{g_1, \cdots, g_n\}$. Although we specialize to the FIR modeling case, all the results apply *mutadis mutandis* to any other affine model of $G(z)$, *e.g.*, Laguerre or Kautz models as described in [5]. The essence of the problem addressed here is, in our opinion, the motivation for the work described in the recent special issue [6] on system identification for robust control design. In comparison with [2], the smoothness parameters M, ρ are not estimated by modeling the tail of the impulse response $\{g_{n+1}, g_{n+2}, \ldots\}$ as a random variable. Our attempt here is to precisely determine the *effect* of the unmodeled dynamics, *i.e.*, the tail of the impulse response, on a least-squares parameter estimator, without any further prior assumptions.

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Least-Squares Estimation

In this section we use least-squares on the measured data to estimate the first n impulse response coefficients $\{g_1, g_2, \ldots\}$ in (2). Towards this end, the unknown impulse response parameters $\{g_1, \ldots, g_L\}$ are partitioned into the (finite) parameter vector to be estimated,

$$\alpha = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} \in \mathbb{R}^n \tag{5}$$

which consists of the first n impulse response coefficients, and the (infinite) parameter vector

$$\beta = \begin{bmatrix} g_{n+1} \\ g_{n+2} \\ \vdots \end{bmatrix} \in \mathbb{R}^\infty \tag{6}$$

which is the remainder of the impulse response. These parameters – the "tail" of the impulse response, $\{g_{n+1}, g_{n+2}, \ldots\}$ – can significantly bias the estimate of the "head," namely, $\{g_1, \ldots, g_n\}$. Statisticians refer to β as a "nuisance" parameter. Note that because G is stable, $\|\beta\|$ is not only finite, but decreases exponentially as n increases. That is, using (3),

$$\|\beta\|^2 = \sum_{k=n+1}^{\infty} g_k^2 \leq \frac{M^2 \rho^{2n}}{1-\rho^2} \tag{7}$$

Using the definition of α and β together with (1) gives,

$$Y = X\alpha + \widetilde{X}\beta + E \tag{8}$$

where

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \in \mathbb{R}^N, \quad E = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix} \in \mathbb{R}^N \tag{9}$$

$$X = \begin{bmatrix} u_0 & \cdots & u_{1-n} \\ \vdots & & \vdots \\ u_{N-1} & \cdots & u_{N-n} \end{bmatrix} \in \mathbb{R}^{N \times n} \tag{10}$$

$$\widetilde{X} = \begin{bmatrix} u_{-n} & u_{-n-1} & \cdots \\ \vdots & & \vdots \\ u_{N-n-1} & u_{N-n-2} & \cdots \end{bmatrix} \in \mathbb{R}^{N \times \infty} \tag{11}$$

Assuming that $X'X \in \mathbb{R}^{n \times n}$ is non-singular, *i.e.*, u is persistently exciting of order n, the least-squares estimate of α is given by the well known formula:

$$\widehat{\alpha} = \begin{bmatrix} \widehat{g}_1 \\ \vdots \\ \widehat{g}_n \end{bmatrix} = \arg \min_{\theta \in \mathbb{R}^n} \|Y - X\theta\|^2 = (X'X)^{-1}X'Y \tag{12}$$

where $\{\widehat{g}_k \mid k = 1:n\}$ can be thought of as estimates of $\{g_k \mid k = 1:n\}$. We also take the estimate of λ_e, the output error variance, as the sample-variance,

$$\widehat{\lambda}_e = \frac{1}{N} \|Y - X\widehat{\alpha}\|^2 \tag{13}$$

When $\beta = 0$, it is well known that $\widehat{\alpha}$ and $\widehat{\lambda}_e$ are the maximum likelihood estimates of α and λ_e, respectively, *e.g.*, [1]. In our case, $\beta \neq 0$, and its effect on the estimates is the subject of the next section.

Statistical Analysis

In this section we analyze the effect of the nuisance parameter β on the estimates $\widehat{\alpha}$ and $\widehat{\lambda}_e$ of α and λ_e, respectively. We use the standard notation $\mathcal{N}(\mu, \Sigma)$ to denote a gaussian distribution with mean μ and variance Σ. Likewise, $\chi^2(m)$ denotes a chi-squared distribution with m degrees of freedom. Recall that if $q \in \mathbb{R}^m$ is drawn from $\mathcal{N}(0, R)$ with R non-singular, then $q'R^{-1}q \in \chi^2(m)$. We also use $\chi^2(m, \tau)$ to denote a *non-central* chi-squared distribution with m degrees of freedom and non-centrality parameter τ. To fix the definition of the non-centrality parameter, if $q \in \mathbb{R}^m$ is drawn from $\mathcal{N}(\mu, R)$, then $q'R^{-1}q \in \chi^2(m, \tau)$ with $\tau = \mu'R^{-1}\mu$. From [3], we also use: as either[1] m or $\tau \to \infty$, $\chi^2(m, \tau) \to \mathcal{N}(m + \tau, 2(m + 2\tau))$. Hence, $\chi^2(m, 0) = \chi^2(m)$ and as $m \to \infty$, $\chi^2(m) \to \mathcal{N}(m, 2m)$.

It is convenient to define the "covariance" matrices,[2]

$$\Sigma_{11} = \frac{1}{N} X'X \in \mathbb{R}^{n \times n} \tag{14}$$

$$\Sigma_{12} = \frac{1}{N} X'\widetilde{X} \in \mathbb{R}^{n \times \infty} \tag{15}$$

$$\Sigma_{22} = \frac{1}{N} \widetilde{X}'\widetilde{X} \in \mathbb{R}^{\infty \times \infty} \tag{16}$$

Observe that only Σ_{11} can be formed from the data and by assumption is invertible.

The following theorem describes the distributions of the key random variables.

Theorem 1 *Define the parameter error,*

$$\widetilde{\alpha} = \widehat{\alpha} - \alpha \tag{17}$$

and the output error,

$$\widehat{E} = Y - X\widehat{\alpha} \tag{18}$$

Under assumptions (a1)-(a4),

(i) The parameter error $\widetilde{\alpha}$ and the residual $\widehat{E}$ are independent and normally distributed as follows:

$$\widetilde{\alpha} \in \mathcal{N}\left(\Sigma_{11}^{-1}\Sigma_{12}\beta, \ \frac{\lambda_e}{N}\Sigma_{11}^{-1}\right) \tag{19}$$

$$\widehat{E} \in \mathcal{N}\left(\Gamma\widetilde{X}\beta, \ \lambda_e \cdot \Gamma\right) \tag{20}$$

where $\Gamma \in \mathbb{R}^{N \times N}$, given by,

$$\Gamma = I_N - X(X'X)^{-1}X' \tag{21}$$

has rank $N - n$ and is idempotent, i.e., $\Gamma = \Gamma^2$.

[1] It can be shown that this result is also true if both m or $\tau \to \infty$.

[2] Although the matrices Σ_{12}, Σ_{22} are infinite dimensional, they always appear multiplying β. Hence, these terms are bounded because the elements in β decay exponentially.

(ii) $\frac{N\widehat{\lambda}_e}{\lambda_e}$ *and* $\frac{N}{\lambda_e}\widetilde{\alpha}'\Sigma_{11}\widetilde{\alpha}$ *have the following non-central chi-squared distributions:*

$$\frac{N\widehat{\lambda}_e}{\lambda_e} \in \chi^2\left(N-n,\ \frac{N}{\lambda_e}\delta\right) \quad (22)$$

$$\frac{N}{\lambda_e}\widetilde{\alpha}'\Sigma_{11}\widetilde{\alpha} \in \chi^2\left(n,\ \frac{N}{\lambda_e}\gamma\right) \quad (23)$$

where

$$\gamma = \beta'\Sigma_{12}'\Sigma_{11}^{-1}\Sigma_{12}\beta \quad (24)$$

$$\delta = \beta'\Sigma_{22}\beta - \gamma = \beta'\left(\Sigma_{22} - \Sigma_{12}'\Sigma_{11}^{-1}\Sigma_{12}\right)\beta \quad (25)$$

(iii) As $N \to \infty$,

$$\widehat{\lambda}_e \to \mathcal{N}\left(\lambda_e+\delta,\ \frac{2\lambda_e}{N}(\lambda_e+2\delta)\right) \quad (26)$$

$$\widetilde{\alpha}'\Sigma_{11}\widetilde{\alpha} \to \mathcal{N}\left(\frac{n\lambda_e}{N}+\gamma,\ \frac{2\lambda_e}{N}\left(\frac{n\lambda_e}{N}+2\gamma\right)\right) \quad (27)$$

The results in part(i) follow directly from the underlying assumptions and definitions of the variables, and except for the non-zero bias terms, are standard, *e.g.*, [1]. Part (ii) is non-standard, in that the error statistics involve non-central chi-square distributions. These results are obtained by direct appeal to the relation between a normally distributed random variable and the non-central chi-squred statistic as stated in the introduction to this section. The asymptotic results in part(iii) follow from the asymptotic normal approximation to a non-central chisquare distribution as stated in the introduction to this section.

In part (iii) of the theorem, the asymptotic variances decay as $1/N$. Hence, for sufficiently large N, the random variable approaches the mean with high probability. This leads directly to the following:

Approximation 1 *For sufficiently large* N, *the following approximations hold with high probability,*

$$\widehat{\lambda}_e \approx \lambda_e + \delta \quad (28)$$

$$\widetilde{\alpha}'\Sigma_{11}\widetilde{\alpha} \approx \frac{n}{N}\lambda_e + \gamma \quad (29)$$

Observe that for large N, the variance estimate $\widehat{\lambda}_e$ tends to over-estimate the true variance λ_e. In addition, the errors $\widetilde{\alpha}$ and $\widehat{\lambda}_e - \lambda_e$ are driven by the "nuisence" parameter β, *i.e.*, the tail of the impulse response.

A special case of interest is when the input u is white, *i.e.*,

$$\Sigma_{11} = \lambda_u \cdot I_n,\ \Sigma_{12} = 0,\ \Sigma_{22} = \lambda_u \cdot I_\infty \quad (30)$$

Theorem 2 *If* u *is white,* i.e., *(30) holds, then:*

$$\frac{N\widehat{\lambda}_e}{\lambda_e} \in \chi^2\left(N-n,\ \frac{N}{\lambda_e}\lambda_u\|\beta\|^2\right) \quad (31)$$

$$\frac{N}{\lambda_e}\lambda_u\|\widetilde{\alpha}\|^2 \in \chi^2(n) \quad (32)$$

In addition, as $N \to \infty$,

$$\widehat{\lambda}_e \to \mathcal{N}\left(\lambda_e+\lambda_u\|\beta\|^2,\ \frac{2\lambda_e}{N}\left(\lambda_e+2\lambda_u\|\beta\|^2\right)\right) \quad (33)$$

The asymptotic part of the above theorem leads to the following:

Approximation 2 *For sufficiently large* N, *if* u *is white,* i.e., *(30) holds, then with high probability:*

$$\widehat{\lambda}_e \approx \lambda_e + \lambda_u\|\beta\|^2 \quad (34)$$

$$\|\widetilde{\alpha}\|^2 \leq \frac{3n}{N}\frac{\lambda_e}{\lambda_u} \quad (35)$$

Large N and High Probability

When the input is white, "large N" can be taken as,

$$N >> \frac{2(1+2\eta)}{(1+\eta)^2},\quad \eta = \frac{\lambda_u\|\beta\|^2}{\lambda_e} \quad (36)$$

where η is the ratio of the energy in the tail to the output error energy. Typical values of N, *e.g.*, 500-1000, will always be well in excess of variations caused by η. Moreover, from central and non-central chi-square tables (*e.g.*, [3]), values of $N \geq 100$ and $n \geq 20$ make the normal approximations very accurate. In consequence, "high probability" is i.. excess of 99.95% for typical data lengths and model orders. Similar numbers hold for the general case with a non-white input.

Frequency Response Estimation

The results of the previous section can be used to analyze the errors in frequency response estimation. Towards this end, express $G(z)$, the true transfer function as,

$$G(z) = D(z)'\alpha + \widetilde{D}(z)'\beta \quad (37)$$

where

$$D(z) = \begin{bmatrix} z^{-1} \\ \vdots \\ z^{-n} \end{bmatrix},\ \widetilde{D}(z) = \begin{bmatrix} z^{-(n+1)} \\ z^{-(n+2)} \\ \vdots \end{bmatrix} \quad (38)$$

Let $\widehat{G}(z)$ denote the transfer function estimate of $G(z)$ defined as

$$\widehat{G}(z) = D(z)'\widehat{\alpha} \quad (39)$$

where $\widehat{\alpha}$ is the least-squares parameter estimate from (12) of the the first n impulse response coefficients of $G(z)$. Let $\Delta(z)$ denote the transfer function error defined as,

$$\Delta(z) = G(z) - \widehat{G}(z) \quad (40)$$

$$= -D(z)'\widetilde{\alpha} + \widetilde{D}(z)'\beta \quad (41)$$

where

$$D(z)'\widetilde{\alpha} = \sum_{k=1}^{n}(\widehat{g}_k - g_k)z^{-k},\quad \widetilde{D}(z)'\beta = \sum_{k=n+1}^{\infty} g_k z^{-k} \quad (42)$$

with $\widetilde{\alpha}$ the parameter error from (17).

From Theorem 1 the following result is obtained.

Theorem 3 *The following results hold at each frequency* ω:

(i) Normal distribution

$$\Delta(e^{j\omega}) \in \mathcal{N}\left(F(e^{j\omega})'\beta,\ \frac{\lambda_e}{N} D(e^{j\omega})^* \Sigma_{11}^{-1} D(e^{j\omega})\right) \quad (43)$$

where

$$F(z)' = \tilde{D}(z)' - D(z)'\Sigma_{11}^{-1}\Sigma_{12} \quad (44)$$

(ii) Non-central chi-squared distribution

$$\frac{|\Delta(e^{j\omega})|^2}{\frac{\lambda_e}{N} D(e^{j\omega})^* \Sigma_{11}^{-1} D(e^{j\omega})} \in \chi^2(1, \epsilon(\omega)) \quad (45)$$

with non-centrality parameter,

$$\epsilon(\omega) = \frac{|F(e^{j\omega})'\beta|^2}{\frac{\lambda_e}{N} D(e^{j\omega})^* \Sigma_{11}^{-1} D(e^{j\omega})} \quad (46)$$

(iii) Asymptotic Normality

As $N \to \infty$,

$$\frac{|\Delta(e^{j\omega})|^2}{\frac{\lambda_e}{N} D(e^{j\omega})^* \Sigma_{11}^{-1} D(e^{j\omega})} \to \mathcal{N}(1 + \epsilon(\omega),\ 2(1 + 2\epsilon(\omega))) \quad (47)$$

Part (iii) leads to the following result.

Approximation 3 *For sufficiently large N, the following approximation holds with high probability at each frequency ω:*

$$|\Delta(e^{j\omega})|^2 \approx \frac{\lambda_e}{N} D(e^{j\omega})^* \Sigma_{11}^{-1} D(e^{j\omega}) + |F(e^{j\omega})'\beta|^2 \quad (48)$$

Observe that if u is white (30) then

$$\begin{aligned} D(e^{j\omega})^* \Sigma_{11}^{-1} D(e^{j\omega}) &= D(e^{j\omega})^* \left(\frac{1}{\lambda_u} I_n\right) D(e^{j\omega}) \\ &= \frac{1}{\lambda_u} D(e^{j\omega})^* D(e^{j\omega}) = \frac{n}{\lambda_u} \end{aligned}$$

This leads to the following:

Theorem 4 *If u is white,* i.e., *(30) holds, then at each frequency ω:*

(i) Normal distribution

$$\Delta(e^{j\omega}) \in \mathcal{N}\left(\tilde{D}(e^{j\omega})'\beta,\ \frac{n}{N}\frac{\lambda_e}{\lambda_u}\right) \quad (49)$$

(ii) Non-central chi-squared distribution

$$\frac{|\Delta(e^{j\omega})|^2}{\left(\frac{n}{N}\frac{\lambda_e}{\lambda_u}\right)} \in \chi^2(1, \epsilon(\omega)) \quad (50)$$

with non-centrality parameter

$$\epsilon(\omega) = \frac{|\tilde{D}(e^{j\omega})'\beta|^2}{\left(\frac{n}{N}\frac{\lambda_e}{\lambda_u}\right)} \quad (51)$$

(iii) Asymptotic Normality

As $N \to \infty$,

$$\frac{|\Delta(e^{j\omega})|^2}{\left(\frac{n}{N}\frac{\lambda_e}{\lambda_u}\right)} \to \mathcal{N}(1 + \epsilon(\omega),\ 2(1 + 2\epsilon(\omega))) \quad (52)$$

Part (iii) together with Approximation 2 leads to:

Approximation 4 *If u is white,* i.e., *(30) holds, then for sufficiently large N, the following approximation holds with high probability at each frequency ω:*

$$|\Delta(e^{j\omega})|^2 \approx \frac{n}{N}\frac{\lambda_e}{\lambda_u} + |\tilde{D}(e^{j\omega})'\beta|^2 \quad (53)$$

Robust Control Analysis

In this section, we use the asymptotic frequency domain bounds to evaluate controller robustness. The goal of control is to reduce the output variance. Consider the LTI feedback controller

$$u = -\hat{K}y \quad (54)$$

where $\hat{K}$ stabilizes the "estimated" FIR system

$$y = \hat{G}u + e,\ \hat{G}(z) = \sum_{k=1}^{n} g_k z^{-k} \quad (55)$$

Applying the control (54) to the actual system (1) yields the closed-loop system

$$y = \frac{\hat{T}}{1 + \hat{Q}\Delta} e,\ u = -\frac{\hat{Q}}{1 + \hat{Q}\Delta} e \quad (56)$$

where

$$\hat{T} = \frac{1}{1 + \hat{G}\hat{K}},\ \hat{Q} = \frac{\hat{K}}{1 + \hat{G}\hat{K}} \quad (57)$$

with Δ the estimation error as defined in (40). Since the nominal system is stable, it follows that Δ, $\hat{T}$, and $\hat{Q}$ are stable transfer functions. Hence, the closed-loop system is stable if and only if,

$$|1 + \hat{Q}(e^{j\omega})\Delta(e^{j\omega})| > 0,\ \forall |\omega| \le \pi \quad (58)$$

If this holds, then the spectrum of y, under closed-loop *-not during identification-* is given by:

$$\Phi_y(\omega) = \left|\frac{\hat{T}(e^{j\omega})}{1 + \hat{Q}(e^{j\omega})\Delta(e^{j\omega})}\right|^2 \lambda_e \quad (59)$$

Suppose that u, *during identification*, is white, i.e., (30) holds. To establish stability, observe that a sufficient condition for stability is that,

$$|\hat{Q}(e^{j\omega})|\ |\Delta(e^{j\omega})| < 1,\ \forall |\omega| \le \pi \quad (60)$$

Using the expression for $|\Delta(e^{j\omega})|$ in Approximation 4 and substituting for λ_e from (34). it follows that for large N, the closed-loop system is stable, with high probability, if,

$$|\widehat{Q}(e^{j\omega})|^2 \left[\frac{3n}{N}\left(\frac{\widehat{\lambda}_e}{\lambda_u} - \|\beta\|^2\right) + |\widetilde{D}(e^{j\omega})'\beta|^2\right] < 1,\ \forall|\omega| \leq \pi \tag{61}$$

Hence, using the large N approximations, with high probability, the output spectrum is bounded as follows:

$$\Phi_y(\omega) \leq \frac{|\widehat{T}(e^{j\omega})|^2\left(\widehat{\lambda}_e - \lambda_u\|\beta\|^2\right)}{\left(1 - |\widehat{Q}(e^{j\omega})|\left[\frac{3n}{N}\left(\frac{\widehat{\lambda}_e}{\lambda_u} - \|\beta\|^2\right) + |\widetilde{D}(e^{j\omega})'\beta|^2\right]^{1/2}\right)^2} \tag{62}$$

The only unknown quantity is β. From (34), we also know with high probability that,

$$\lambda_e \approx \widehat{\lambda}_e - \lambda_u\|\beta\|^2$$

Since λ_e must be positive, it follows that

$$\|\beta\|^2 \leq \widehat{\lambda}_e/\lambda_u \tag{63}$$

provides a worst-case upper bound. Observe that this bound is known because $\widehat{\lambda}_e$ is the computed variance estimate and λ_u is selected by the user as the input variance. As a practical matter, it is unlikely that β will achieve this bound. If it did, then the noise variance $\lambda_e \approx 0$, which for large N, will almost never occur.

Using (3), we get

$$|\widetilde{D}(e^{j\omega})'\beta| = |\sum_{k=n+1}^{\infty} g_k e^{-j\omega k}| \leq \frac{M\rho^n}{1-\rho}$$

Hence, for large N, the closed-loop system is stable with high probability if,

$$|\widehat{Q}(e^{j\omega})|^2\left[\frac{3n}{N}\frac{\widehat{\lambda}_e}{\lambda_u} + \frac{M^2\rho^{2n}}{(1-\rho)^2}\right] < 1,\ \forall|\omega| \leq \pi \tag{64}$$

The constants M and ρ are unknown, so in order to evaluate the above robustness condition, either we require *a priori* knowledge or infer the values from the first n impulse response coefficients $\widehat{\alpha}' = [\widehat{g}_1 \cdots \widehat{g}_n]$. That is, define the estimates $\widehat{M}, \widehat{\rho}$ via

$$|\widehat{g}_k| \leq \widehat{M}\widehat{\rho}^{k-1},\ \forall k \in [1, n] \tag{65}$$

and replace M, ρ with $\widehat{M}, \widehat{\rho}$. This leads to the robustness test:

$$|\widehat{Q}(e^{j\omega})|^2\left[\frac{3n}{N}\frac{\widehat{\lambda}_e}{\lambda_u} + \frac{\widehat{M}^2\widehat{\rho}^{2n}}{(1-\widehat{\rho})^2}\right] < 1,\ \forall|\omega| \leq \pi \tag{66}$$

Now, suppose that the closed-loop system is stable and the above inequality holds. Then the spectrum of y is bounded, with high probability, by: of y and u are given, respectively, by:

$$\Phi_y(\omega) \leq \frac{|\widehat{T}(e^{j\omega})|^2\widehat{\lambda}_e}{\left(1 - |\widehat{Q}(e^{j\omega})|\left[\frac{n}{N}\frac{\widehat{\lambda}_e}{\lambda_u} + \frac{\widehat{M}^2\widehat{\rho}^{2n}}{(1-\widehat{\rho})^2}\right]^{1/2}\right)^2} \tag{67}$$

The above bound gives an indication of the trade between bias and variance as the model order varies – all results being valid for data length $N \geq 500$ with probability in excess of 99.95

Concluding Remarks

Using an output error linear plant, we have shown that with gaussian noise and affine models, there is a very rich structure in the analysis of standard least-squares estimation of the first n impulse response coefficients. The remaining coefficients bias the estimate in a precisely defined way involving non-central chi-squared statistics. These appear to be extremely useful in predicting model error for robust control design from finite data records. Much still remains to be done even for this restricted and analytically tractable case, particularly in finding a means to bound the effect of the bias (the tail of the impulse response) without having to perform additional identification with ever larger parameter orders. This ultimately may involve additional *a priori* quantitative knowlwdge. We feel that this paper indicates a first step towards the more difficult problem of model structures which account for non-white noise, *e.g.*, ARX or ARMAX models.

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Worst-Case Control Design from Batch-Least-Squares Identification

ROBERT L. KOSUT HENK ALING *†

Abstract A case study is presented to support the thesis that high order models obtained from batch-least-squares provide all the necessary significant information for robust control design.

1 Introduction

Suppose that the measured sampled-data set

$$\{y_t, u_t \mid t = 1 : N\} \tag{1}$$

has been obtained from an unknown system where u is the scalar control input and y is the scalar sensed output. Suppose further that it is known *a priori* that the system which generated the data is stable, linear-time-invariant (lti), and operating in open-loop. Hence,

$$y = Gu + v \tag{2}$$

where G is an lti operator with unknown transfer function $G(z)$. The output disturbance v_t is known to be a zero-mean sequence with the unknown spectrum $\Phi_v(\omega)$. The problem is to use the measured data set (1) together with the *a priori* information to obtain estimates $\widehat{G}(z)$ and $\widehat{\Phi}_v(\omega)$ which can be used for control design.

To see the control problem more clearly, suppose somehow we have obtained estimates $\widehat{G}(z)$ and $\widehat{\Phi}_v(\omega)$. The next step is to design a feedback controller. Let the control be

$$u = -Ky \tag{3}$$

where K is lti with transfer function $K(z)$. Typically the controller K is designed for the estimated system

$$y = \widehat{G}u + v, \quad \text{spectrum}\{v\} = \widehat{\Phi}_v(\omega) \tag{4}$$

The problem is that the estimated system differs from the true system (2) and hence, predicted performance, based on the estimate, may not at all be like the performance actualized when the controller is applied. If a bound on the model error between the true and estimated systems were known, then a robust controller could be designed. Various approaches have been put forth to resolve this problem, *e.g.*, [3], but these involve forms of prior information and/or approximations which are either very difficult to obtain or are too coarse. For control design, the error needs to be well known near the desired bandwidth of the closed-loop system, which may not be known beforehand. Hence, prior information on the impulse response, such as magnitude and rate of decay, is unlikely to contribute significantly to a useable estimate of the error near the desired bandwidth because the impulse response bound provides only very low frequency and very high frequency information. Ironically, any precise information about the system dynamics near the desired bandwidth is likely to preclude the need for identification.

In this paper we propose the thesis that high order models obtained from batch-least-squares provide all the necessary significant information for robust control design *without invoking additional prior information.* A case study is presented which (of course) supports the thesis.

2 Batch-Least-Squares

Perhaps the most widely used procedure for obtaining the estimates is via *batch-least-squares* where:

$$\widehat{G}(z) = \frac{\widehat{B}(z)}{\widehat{A}(z)} = \frac{\widehat{b}_1 z^{-1} + \cdots + \widehat{b}_n z^{-n}}{1 + \widehat{a}_1 z^{-1} + \cdots + \widehat{a}_n z^{-n}} \tag{5}$$

$$\widehat{\Phi}_v(\omega) = \frac{\widehat{\lambda}}{|\widehat{A}(e^{j\omega})|^2} \tag{6}$$

$$\widehat{\lambda} = \frac{1}{N}\sum_{t=1}^{N}\left[(\widehat{A}y - \widehat{B}u)_t\right]^2 \tag{7}$$

where

$$\widehat{A}(z), \widehat{B}(z) = \arg \min_{A(z),B(z)} \frac{1}{N}\sum_{t=1}^{N}[(Ay - Bu)_t]^2 \tag{8}$$

The number n will be referred to as the *model order.* (Actually the numerator and denominator orders need not be the same as shown here.) In every practical situation

*The authors are with Integrated Systems, Inc., 3260 Jay Street, Santa Clara, CA 95054.

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there is no finite value of n for which the right hand side in (8) is zero. That is, *the true system is not in the model set* - an axiom.

The great appeal of "least-squares," and the principle reason for its ubiquity, is that, provided the input is sufficiently rich in spectral content, a unique minimum of (8) is always obtained. Furthermore, there are very efficient and reliable methods for computing the solution, typically involving square-root calculations such as the QR transformation as well as lattice forms for very high model orders. It is imperative that the calculations are done in this manner, for otherwise significant numerical errors will accrue, even for a small number of parameters. There are other reasons as well for using a QR method, *e.g.*, (1) high model orders and large amounts of data are easily handled, (2) data from different experiments are readily combined without re-doing the entire estimation, and (3) prediction errors can be computed for varying model orders directly from the QR transformation. These factors make it possible to easily and rapidly generate extremely high order models from large amounts of data. This facility in turn provides a great deal of information about the true system,

3 Robust Control Design

Using the control (3) on the plant (2) yields the closed-loop system

$$\begin{aligned} y &= T_{yv} v \\ u &= -T_{uv} v \end{aligned} \tag{9}$$

where

$$T_{yv} = \frac{1}{1+GK}\ , \quad T_{uv} = \frac{K}{1+GK} \tag{10}$$

To arrive at an expression involving the estimates, let

$$\varepsilon = \hat{A}y - \hat{B}u \tag{11}$$

denote the *prediction error* after identification. Using the plant description (2) gives the equivalent expression for ε:

$$\begin{aligned} \varepsilon &= w + \Delta u \\ w &= \hat{A}v \\ \Delta &= \hat{A}G - \hat{B} \end{aligned} \tag{12}$$

As a result the closed-loop system is equivalently:

$$\begin{aligned} y &= T_{yw} w \\ u &= -T_{uw} w \end{aligned} \tag{13}$$

where

$$T_{yw} = \frac{\hat{T}_{y\varepsilon}}{1+\Delta\hat{T}_{u\varepsilon}}\ , \quad T_{uw} = \frac{\hat{T}_{u\varepsilon}}{1+\Delta\hat{T}_{u\varepsilon}} \tag{14}$$

and

$$\hat{T}_{y\varepsilon} = \frac{\hat{H}}{1+\hat{G}K}\ , \quad \hat{T}_{u\varepsilon} = \frac{\hat{H}K}{1+\hat{G}K}\ , \quad \hat{H} = \frac{1}{\hat{A}} \tag{15}$$

Since K is designed for the estimated system, it follows that $\hat{T}_{y\varepsilon}$ and $\hat{T}_{u\varepsilon}$ are stable. Hence, K stabilizes the true system if and only if $(1+\Delta\hat{T}_{u\varepsilon})^{-1}$ is stable. Because both $\hat{T}_{u\varepsilon}$ and Δ are stable, K stabilizes the true system if and only if

$$|1 + \Delta(e^{j\omega})\hat{T}_{u\varepsilon}(e^{j\omega})| \neq 0, \quad \forall|\omega| \leq \pi \tag{16}$$

The well known condition for robust stability [1], and sufficient for (16), is that the loop-gain be less than one, *i.e.*,

$$|\hat{T}_{u\varepsilon}(e^{j\omega})\Delta(e^{j\omega})| < 1, \quad \forall|\omega| \leq \pi \tag{17}$$

To verify either (16) or (17) requires some means of estimating $\Delta(e^{j\omega})$ or a bound on $|\Delta(e^{j\omega})|$. In addition, to predict closed-loop performance requires producing an estimate of $\Phi_w(\omega)$, the sprectrum of w as defined in (12). Estimates of both can be obtained using standard spectral methods (Ch.6,[2]) as follows:

$$\begin{aligned} \hat{\Delta}(\omega) &= \hat{\Phi}_{\varepsilon u}(\omega)/\hat{\Phi}_u(\omega) \\ \hat{\Phi}_w(\omega) &= \hat{\Phi}_\varepsilon(\omega) - |\hat{\Phi}_{\varepsilon u}(\omega)|^2/\hat{\Phi}_u(\omega) \end{aligned} \tag{18}$$

All the $\hat{\Phi}$ variables are generated from the post-identification data set:

$$\{\varepsilon_t, u_t \mid t = 1:N\} \tag{19}$$

It is important to mention that spectral estimation techniques also introduce errors. How the spectral estimate varies from the true is not known precisely although asymptotic results are avaialble [2]. These are similar to asymptotic results for estimating model error from batch-least-squares. Unfortunately, precise conditions for which the asymptotic results are good approximations are not known without invoking additional prior information, which we argue may not be obtainable in practice. For this reason we procede heuristically, and simply utilize the spectral estimates.

Based on these estimates, we obtain the following approximations to the *closed-loop* rms values

$$\begin{aligned} \text{rms}(y) &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} \left|\hat{T}_{yw}(e^{j\omega})\right|^2 \hat{\Phi}_w(\omega)d\omega \\ \text{rms}(u) &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} \left|\hat{T}_{uw}(e^{j\omega})\right|^2 \hat{\Phi}_w(\omega)d\omega \end{aligned} \tag{20}$$

where

$$\hat{T}_{yw} = \frac{\hat{T}_{y\varepsilon}}{1+\Delta\hat{T}_{u\varepsilon}}\ , \quad \hat{T}_{uw} = \frac{\hat{T}_{u\varepsilon}}{1+\Delta\hat{T}_{u\varepsilon}} \tag{21}$$

are estimates of the actual closed-loop transfer functions in (14). Following (17), $\widehat{T}_{yw}$ and $\widehat{T}_{uw}$ are stable if the *estimated* loop-gain satisfies,

$$|\widehat{T}_{u\epsilon}(e^{j\omega})\widehat{\Delta}(e^{j\omega})| < 1, \quad \forall|\omega| \leq \pi \tag{22}$$

Since $\widehat{\Delta}(e^{j\omega})$ is an approximation to $\Delta(e^{j\omega})$, satisfaction of (22) does not imply that (17) holds. Moreover, since (17) is sufficient to insure (16), failure of (22) to hold does not imply instability, but certainly merits some caution at those frequencies where the test fails.

4 Case Study

Simulated System The computer simulated (true) system is the mass-spring-damper mechanism shown in figure 1, where u is the control force and d is an exogenous disturbance force. The unknown disturbance d is a zero-mean random sequence with variance $(.001)^2$. The user applies a zero-mean sequence with unit variance as the control input for identification. The data set is stored for 1024 samples. The sampling frequency for both control and sensing is 10 hz.

Model Order Selection Batch-least-squares estimates are computed using MATRIX$_X$ from the data set $\{y_t, u_t \mid t = 1:512\}$, thus, $N = 512$ in (1). The remaining data set, $\{y_t, u_t \mid t = 513:1024\}$, is used for validation. Figure 2 shows the normalized rms values of the predition error for model orders from 1 to 60 on both the identification and validation data sets. Using the identification data, the rms values continually decrease as the order increases, which is to be expected because after a certain point the model is fitting noise. This is verified using the validation data (lower bar plot of figure 2) where the rms values actual begin to slightly *increase* with increasing model order. Thus, beyond the range from $n = 10$ to $n = 20$, no new information is really obtained in the identification. Hence, the "optimal" model order from this data set is in this range. This phenomena can also be seen by examing the prediction error time series, shown in figure 3 for model orders $n = 4, 16, 60$.

In figure 3, as the model order increases from 4 to 16, the variation (rms) of the error decreases. However, increasing the order from 16 to 60 decreases the error over the identification samples ($t = 1:512$), whereas the error slightly increases over the validation samples ($t = 513:1024$). To emphasize the effect of noise fitting, we repeated the experiment with the shorter identification set $\{y_t, u_t \mid t = 1:256\}$. The results are shown in figures 4-5. Now the rms of the prediction error using the validation data increases more sharply for $n > 12$, and the graph of $\epsilon(t)$ for $n = 60$ shows a significantly smaller variation over the identification samples $t = 1:256$.

Control Design Based on the above results it seems reasonable to select a design model with an order in the range $10 \leq n \leq 20$. For illustrative purposes here we select three values $n = 4, 16, 40$. Figure 6 shows the magnitude and phase of the frequency responses of the true system $G(e^{j\omega})$ and the three estimates $\widehat{G}_n(e^{j\omega})$ corresponding to $n = 4, 16, 40$. As suggested by the rms plots in figure 2, the largest errors occur for $n = 4$, and for $n = 40$, the estimates are "noisy." Similarly, figure 7 shows the true spectrum $\Phi_v(\omega)$ and the three spectral estimates $\widehat{\Phi}_{v,n}(\omega)$ corresponding to $n = 4, 16, 40$.

To evaluate the efficacy of the closed-loop rms appoximations in (20), a set of LQG controllers were designed for each of the plant models as follows. For each $n = 4, 16, 40$, the observer was based on the model,

$$\widehat{A}_n y = \widehat{B}_n u + e,$$

where e_t is taken as a white noise with unit variance. The regulator is then designed to minimize the expected value of $\sum_{t=1}^{\infty}[y_t^2 + (\rho u_t)^2]$ for control weights $\rho = 10, 1, .1, .01$. Thus, we obtain the family of 12 controllers,

$$u = -K_{n,\rho} y$$

Figure 8 shows the predicted and actual performance tradeoff between rms(y) and rms(u). Observe that for $n = 16, 40$, the predicted performance is very similar to the actual performance, whereas for $n = 4$, the actual performance is significantly better than the predicted. Recall that $n = 16$ is considered to be an optimal choice based on the cross validation plots in figure 2. Beyond $n = 16$, no significant performance increases were observed.

The performance tradeoff of the different controllers is not at all complete by just examining figure 8. This does not show the robustness properties of the different controllers. For $n = 16$, figure 9 shows the estimated loop-gain $|\widehat{T}_{u\epsilon}(e^{j\omega})\widehat{\Delta}(e^{j\omega})|$ and the actual loop-gain $|\widehat{T}_{e\epsilon}(e^{j\omega})\Delta(e^{j\omega})|$ for two of the 12 LQG designs, namely, for model order $n = 16$ with control weights $\rho = 1, .01$. For the smaller weight, $\rho = .01$, there is no robustness guarantee because the estimated loop gain is greater than 1 at some high frequencies. However, the actual loop gain remains less than 1 for all frequencies. In addition, in every other case (not shown here), the estimated loop-gain was always larger (more conservative) than the actual gain.

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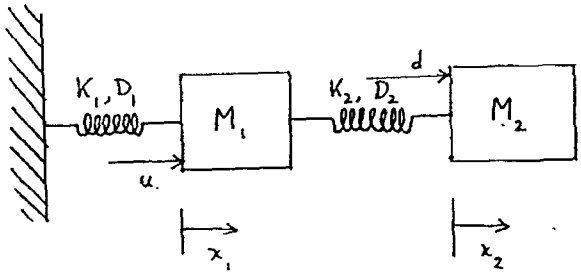

Figure 1: Mass-spring system with $M_1 = 1, M_2 = .25, K_1 = 6, K_2 = .125, D_1 = .3, D_2 = .05$. The sensor reads $y = 100x_2$.

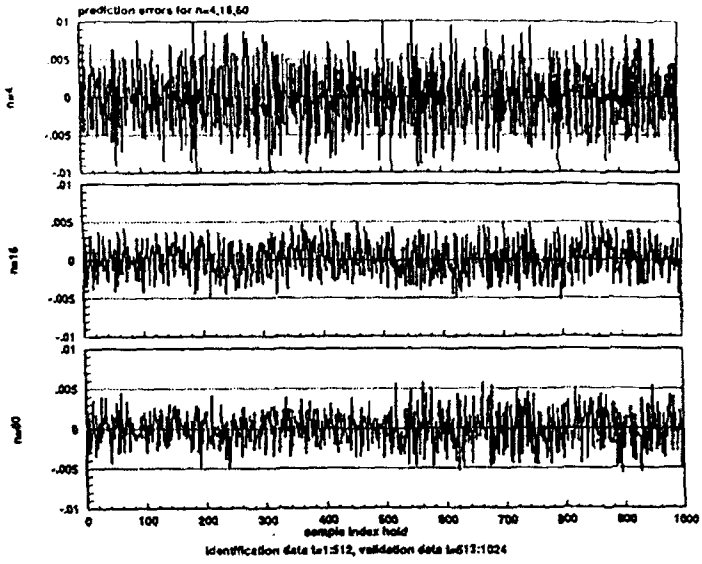

Figure 3: Prediction error $\varepsilon(t)$ for $n = 4, 16, 60$; identification from samples $t = 1 : 512$, validation from samples $t = 513 : 1024$.

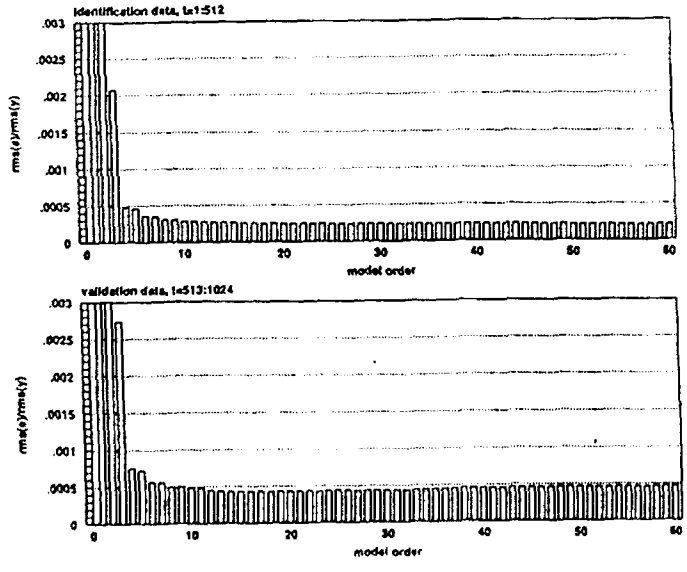

Figure 2: Normalized RMS prediction error vs. model order. Identification for t=1:512, validation for t=513:1024.

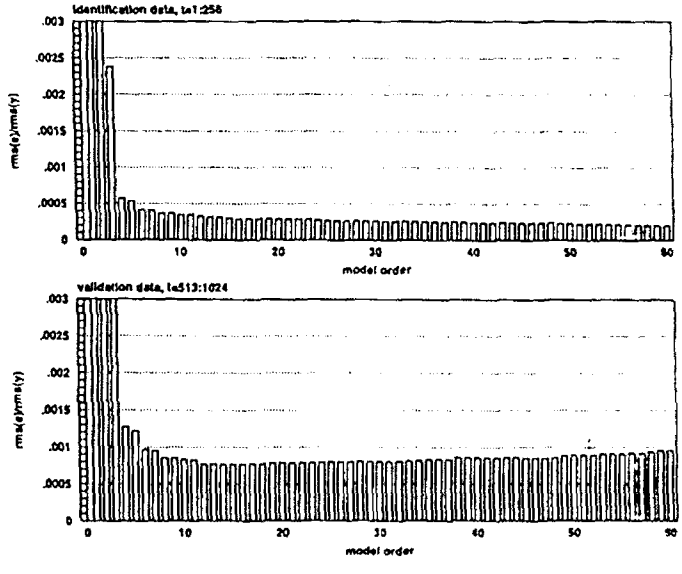

Figure 4: Normalized RMS prediction error vs. model order. Identification for t=1:256, validation for t=513:1024.

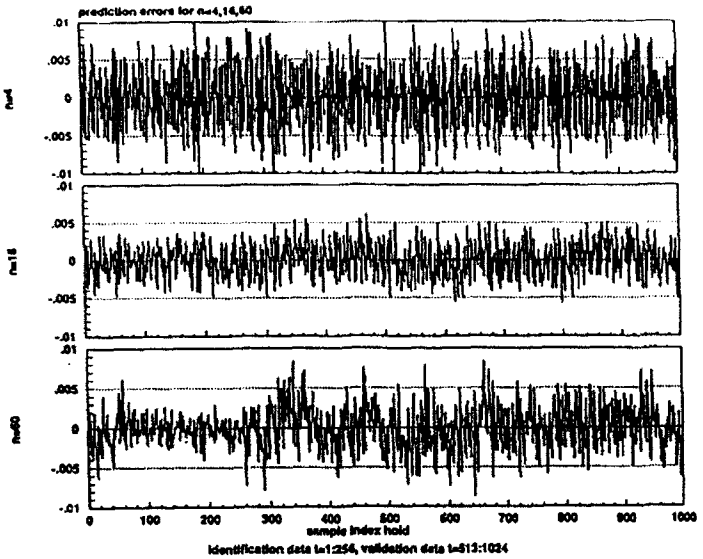

Figure 5: Prediction error $\varepsilon(t)$ for $n = 4, 16, 60$; identification from samples $t = 1 : 256$, validation from samples $t = 513 : 1024$.

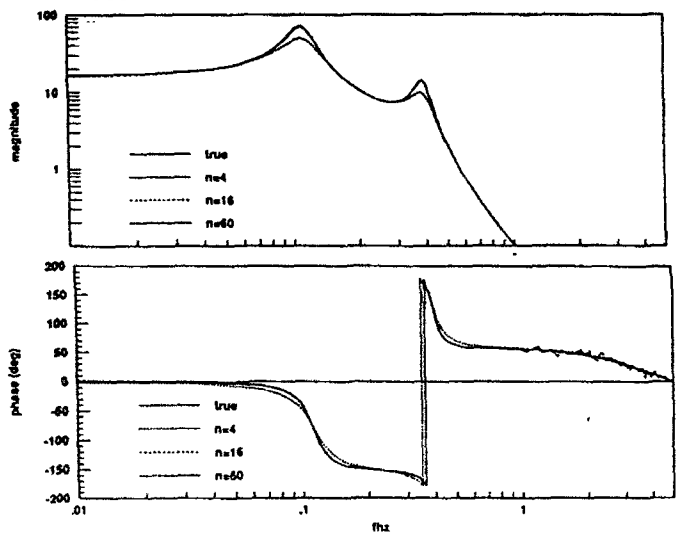

Figure 6: Magnitude and phase plots of true and estimated $G(e^{j\omega})$ for $n = 4, 16, 40$.

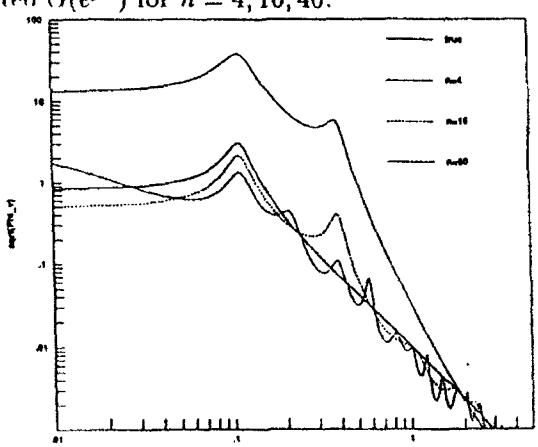

Figure 7: Plots of true and estimated $\Phi_v(\omega)$ for $n = 4, 16, 40$.

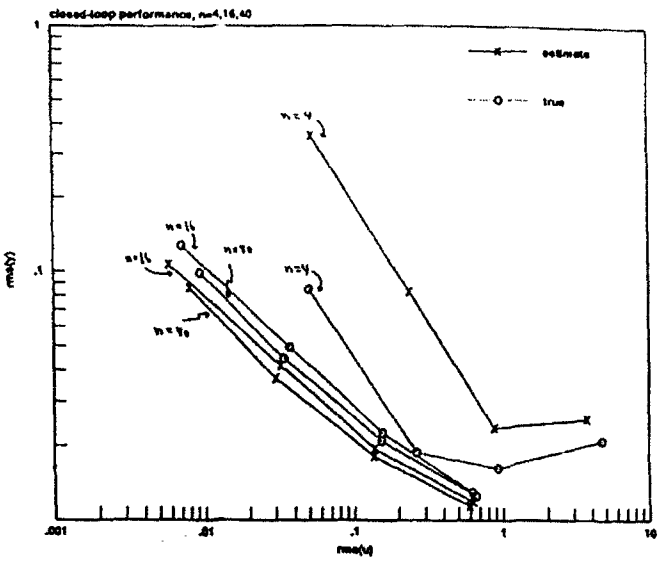

Figure 8: Predicted and actual RMS performance tradeoff for $n = 4, 16, 40$.

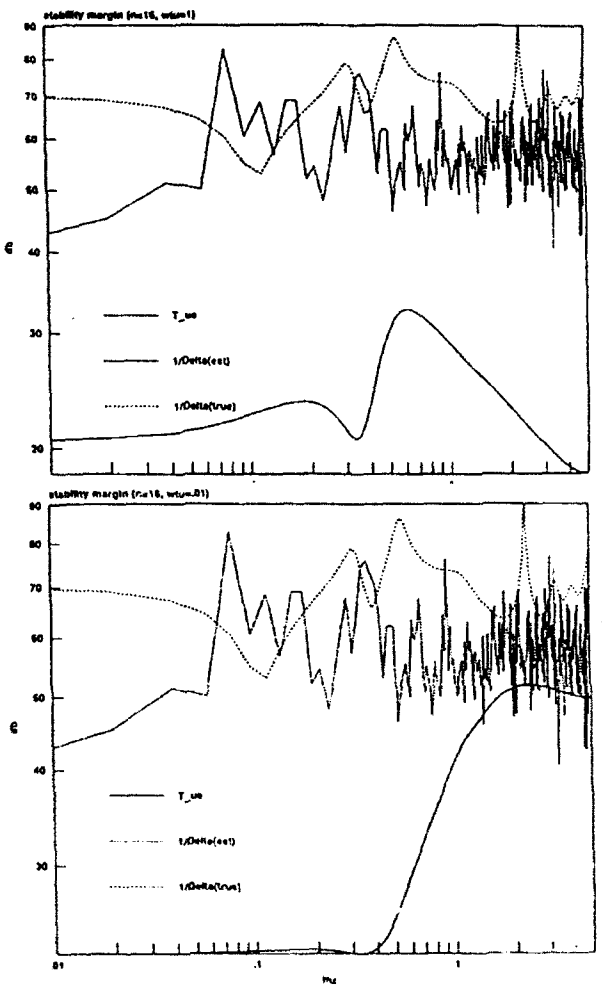

Figure 9: Predicted and actual robustness margins.

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System Identification for Robust Control Design

ROBERT L. KOSUT *†

Abstract Some recent results are summarized in parameter set estimation for linear-time-invariant systems. The extension to nonlinear uncertain systems is explored and some preliminary results are presented. Robust control design requirements are also discussed.

1 Introduction

There are many ways to design or configure an adaptive control system. Figure 1 depicts the self-tuning-regulator (STR) configuration [2]. Two feedback processes make it adaptive, namely: (i) a model parameter estimator, and (ii) a control design rule.

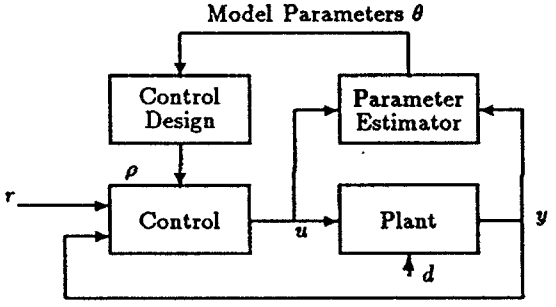

Figure 1: Self Tuning Regulator (STR).

The parameter estimator operates on the input-output data obtained from measurements (y, u) of the plant system and produces a model parameter estimate $\theta \in \mathbb{R}^p$. The parameter estimate is transformed by the control design rule into a controller parameter $\rho \in \mathbb{R}^\ell$, which is then used in a predetermined parametric controller structure in feedback with the actual system.

It is obviously very easy to construct an adaptive system: just connect a control design rule and an estimator together. However, it is very difficult to insure that the resulting adaptive system will provide acceptable performance. This has been the goal of research in this area for over 30 years.

Roughly, if the true system is in the model set which underlies the parameter estimator, then the adaptive system will asymptotically reduce the error signal for arbitrary bounded exogenous inputs (r, d). Technically it is necessary that a a certain (closed-loop) transfer function is strictly-poitive-real (SPR) [18,10,1], *e.g.* , $H(s)$ is SPR if it is stable and satisfies,

$$\mathrm{Re}[H(j\omega)] \geq 0, \forall \omega \qquad (1)$$

The main difficulty, to put it simply, is that the true system is *never* in the model set - there are always dynamical phenomena which remain unaccounted - and unfortunately, the SPR condition fails to hold. Moreover, the theory based on this property is sufficient and hence does not predict what will happen if the SPR condition is violated.

Under sufficiently slow adaptation the method of averaging can be applied to expose a mechanism for stability and instability [1],[21],[2]. This theory replaces the above SPR condition with a "signal dependent positivity condition" of the form,

$$R = \int \mathrm{Re}[H(j\omega)]S(\omega)\, d\omega > 0 \qquad (2)$$

where $S(\omega) > 0$ is a spectral density matrix associated with the exogenous inputs. This condition is much less restrictive because even if $H(j\omega)$ fails to satisfy the SPR condition (1) at some frequencies, (2) can still hold provided the excitation is concentrated at those frequencies where $\mathrm{Re}[H(j\omega)] > 0$. Moreover, if any eigenvalue of R is negative then the system is unstable. In using the theory for design, the user must select an appropriate combination of data filtering and excitation spectrum. This task is similar to problems encountered in system identification [17] except that here the system being identified is in closed-loop, which vastly complicates the selction process.

To see this more clearly, consider the function $\Gamma(\theta)$ defined via Figure 2, *i.e.* , for every parameter choice θ there is a resulting parameter estimate denoted by the function $\Gamma(\theta)$.

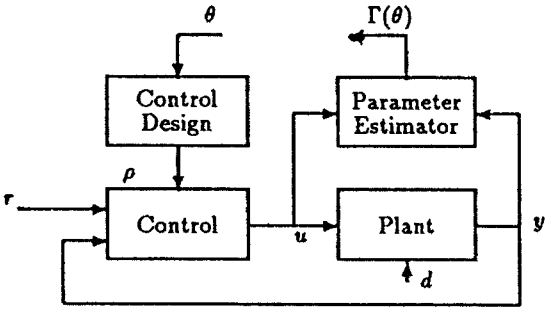

Figure 2: Illustration of the parameter map Γ.

It is shown in [19] that under slow adaptation, convergence points of the STR system in Figure 1 are precisely the fixed-points of Moreover, the fixed-point is stable if (2) holds and is unstable otherwise.

*Senior Scientist, Integrated Systems Inc., 2500 Mission College Blvd., Santa Clara, CA 95054; Consulting Professor, Information Systems Lab, Stanford University.

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In summary, the averaging result shows that stability of the (nonlinear) adaptive system can be deduced from a frequency domain condition (2) which mixes signals and systems. However, there are some difficulties in utilizing the theory. In the first place, it is no trivial task to determine the fixed point(s) of the map Γ, *i.e.* , those $\theta \in \mathbb{R}^p$ satisfying $\theta = \Gamma(\theta)$. Secondly, both the transfer function $H(s)$ and the spectrum $S(\omega)$ depend in a complicated manner on the fixed-point and it is unclear how to precisely manipulate data filters and input spectrum to acheive either a satisfactory fixed-point and/or a satsfactory transient response in the adaptive parameter trajectory. To put it bluntly, the theory fails to produce a "user friendly" design method.

If we agree that the fundamantal difficulty in analyzing the adaptive system is the ubiquitous model uncertainty, then one alternate approach is to configure an adaptive control system which specifically accounts for the uncertainty. One such scheme, depicted in Figure 3, replaces the parameter estimator in Figure 1 with an estimator that produces a model set or set of uncertainty. This would avoid the major obstacle, namely, that the true system is not in the model set used for identification. This type of estimator is referred to as an *uncertainty estimator* or a *set estimator*. This differs from the estimator in the usual adaptive schemes (*cf.* Figure 1), where a *single* estimated model is produced, with no information regarding its accuracy.

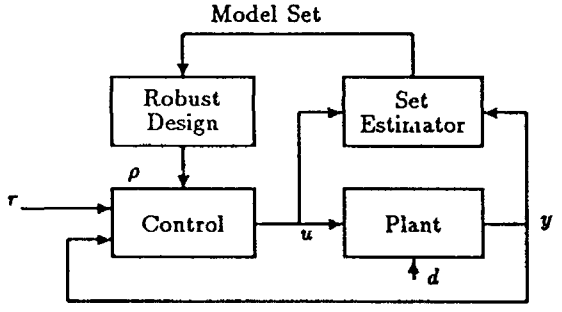

Figure 3: Adaptive control with uncertainty estimation.

The second change is to use a robust control design rule, *i.e.* , one that accepts a model set in the form produced by the set estimator. Under these conditions, if the true system which generated the measured data is contained in the estimated set, then the adaptive system is not only stable, but acheives the maximum performance possible given the estimated set of uncertainty.

Proceding in this way we have transformed the problem of adaptive control design from analysis with trial-and-error into separate synthesis problems in set estimation and robust control design. In effect this is a "separation principal" analogous to that in the LQG design.

At present, methodologies for the design of set estimators are under development, *e.g.* , [23], [12], [16], [13],[9],[14], [24]. On the other hand, there is a reasonable maturity of methodologies for robust control design, particularly for plants with uncertain nonparametric linear dynamics, *e.g.* , [20], [25], [5,6], [8], [22]. Robust control design of plants with parametric uncertainty seems still underdeveloped despite some heroic efforts, *e.g.* , see [3,4] and the references therein.

In the remainder of the paper we principally address set estimation for linear and nonlinear systems. Section 2 provides a review of some recent results in linear set estimation and some new results in nonlinear set estimation. Section 3 provides a brief section on linear robust control of plants with both parametric and nonlinear uncertainty set descriptions.

2 Set Estimation

Set estimators should at least have the following features:

- *Uncertain Parameters.* A capability to account for that part of the system which is known to be governed by physical laws or able to be described by known functions dependent on certain constant parameters. The parameters may only be known to lie within some range of variation.
- *Uncertain Dynamics.* Able to account for uncertain dynamics for which a parametric structure is not available or assumed, *e.g.* , neglected high frequency flexible modes, uncertain memoryless nonlinearities, *etc.*.

2.1 Linear Set Estimation

Consider the linear-time-invariant model set:

$$\mathcal{G}(\Theta, W) = \{G_\theta(1 + \Delta W) : \theta \in \Theta,\ \|\Delta\|_\infty \leq 1\} \qquad (3)$$

The set $\mathcal{G}(\Theta, W)$ describes both parametric and nonparametric uncertainty. The parametric uncertainty is reflected in the set $\{G_\theta : \theta \in \Theta\}$ where G_θ is a *parametric transfer function* with uncertain parameters $\theta \in \Theta \subset \mathbb{R}^p$. The mapping $\theta \rightarrow G_\theta$ is known but the exact parameter values are known only to be in some set Θ. The nonparametric uncertainty is reflected in the set $\{\Delta : \|\Delta\|_\infty \leq 1\}$. Thus Δ is an uncertain linear-time-invariant system only known to be stable and unity bounded in the $\mathcal{H}_\infty$-norm, which for continuous time systems is defined as $\|\Delta\|_\infty = \sup_{\omega \in \mathbb{R}} |\Delta(j\omega)|$ and for discrete time systems as $\|\Delta\|_\infty = \sup_{|\omega| \leq \pi} |\Delta(e^{j\omega})|$. W is a stable transfer function which reflects the size of the relative (or multiplicative) uncertainty, *i.e.* ,

$$\|\Delta\|_\infty = \|\frac{G - G_\theta}{G_\theta W}\|_\infty \leq 1$$

The above expression suggests interpreting the set $\mathcal{G}(\Theta, W)$ as a set of transfer functions "centered" at the parametric transfer function G_θ with a "radius of uncertainty" of $G_\theta W$.

It is usually possible in a modeling process to arrive at an initial parameter set Θ_0 and a weighting transfer function W_0. In the case when the prior set $\mathcal{G}(\Theta_0, W_0)$ is too coarse to lead to tolerable closed-loop performance levels, then a model set estimator is required to refine the prior information by making use of measured data. Specifically, we extract the following result from [14].

THEOREM

Suppose that the the measured data set

$$\{y, u : t = 1, \ldots, N\}$$

is obtained from the sampled-data system

$$y = Gu$$

where G has the discrete-time transfer function $G(z)$. Furthermore, suppose that from prior information

$$G \in \mathcal{G}(\Theta_0, W_0)$$

and the parametric transfer function in (3) has the structure:

$$G_\theta(z) = \frac{B_\theta(z)}{A_\theta(z)} = \frac{b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$
$$\theta^T = [a_1 \cdots a_n \; b_1 \cdots b_m]$$

Under these conditions, if G is initially at rest, and is either stable or in a stabilizing feedback, then:

$$G \in \mathcal{G}(\Theta_0, W_0) \cap \mathcal{G}(\Theta_N, W_0) \qquad (4)$$

where Θ_N is the parameter set estimate,

$$\Theta_N = \{\theta : \|A_\theta y - B_\theta u\|_N \leq \|B_\theta W_0 u\|_N\} \qquad (5)$$

with the N-point signal norm $\|x\|_N^2 = \sum_{t=1}^N x(t)^2$.

The above result implies that if the true parametric transfer function is $G_{\theta_{\text{true}}}$, where $\theta_{\text{true}} \in \mathbb{R}^p$ is the true parameter value, then

$$\theta_{\text{true}} \in \Theta_0 \cap \Theta_N$$

A good data set would insure that the new set estimate is strictly inside the prior estimate, that is

$$\Theta_0 \cap \Theta_N \subset \Theta_0$$

Since both A_θ and B_θ are affine functions of θ, it can be shown [14] that Θ_N describes either an ellipsoid or an hyperboloid in $\mathbb{R}^p$, depending on the data. Moreover, although the set $\Theta_0 \cap \Theta_N$ is not an ellipsoid, nonetheless a bounding ellipsoid can be obtained.

A similar result is obtained in [24] for a co-prime factor nonparametric uncertainty structure rather than the multiplicative one used here. More on bounding ellipsoids can be found in [7] who considered the problem of parameter set estimation with bounded noise and no unmodeled dynamics.

2.2 Nonlinear Set Estimation

The preceding principals of set estimation for linear-time-invariant systems can be applied to the set estimation of nonlinear systems. We will illustrate the problems using the following three example systems: (i) an input nonlinearity, (ii) an output nonlinearity, and (iii) a mechanical system with backlash.

Example 1: Input Nonlinearity

Consider the system shown in Figure 4 and described by:

$$y = G_\theta \bar{u}, \quad \bar{u} = f(u) \qquad (6)$$

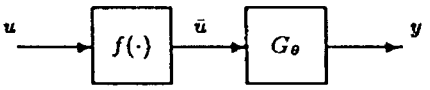

Figure 4: Input nonlinearity.

Make the following assumptions:

1. The function $f(\cdot)$ is a memoryless time-invariant nonlinearity known to lie in the sector

$$|f(u) - ku| \leq \delta|u|, \quad \forall |u| \leq \rho \qquad (7)$$

where $\delta < k$ and $\rho > 0$ are known constants.

2. G_θ is a continuous-time linear-time-invariant system with stable transfer function $G_\theta(s)$ and where $\theta \in \mathbb{R}^p$ are uncertain parameters.

3. The measured data set is

$$\{y(t), u(t) \; : \; t = 1, \ldots, N\}$$

where the time t is normalized to the sampling interval.

The constants (k, δ, ρ) quantify the uncertainty in the nonlinear function $f(\cdot)$ in much the same way that W bounds the uncertain linear-time-invariant nonparametric dynamics in the previous section. A problem here, though, is that $\bar{u}$, the input to the linear part of the system, is not a measured variable. Moreover, the nonlinear function precludes describing any discrete-time transfer function from u into y. However, provided $f(\cdot)$ is sufficiently smooth, for fast sampling we have the following sampled-data approximation

$$y \approx G_\theta \bar{u}, \quad \bar{u} = f(u) \qquad (8)$$

where now $G_\theta(s)$ is approximated by the zero-order-hold z-transform

$$G_\theta(z) = (1 - z^{-1})\mathcal{Z}\{\frac{1}{s} G_\theta(s)\}$$

This approximation is only valid at the sample times $t \in \{1, \ldots, N\}$. For example, if $f(\cdot)$ is a polynomial or rational function, then there certainly exists a (not necessarily small) region $|u| \leq \rho$ such that (7) holds.

To illustrate the problems in obtaining a set estimator even for the approximate system (8), suppose that (k, δ, ρ) are known, and we wish to estimate a parametric model for $G_\theta(z)$. For illustrative purposes, suppose that $G_\theta(z)$ is in the two-parameter set:

$$G_\theta(z) = \frac{B_\theta(z)}{A_\theta(z)} = \frac{bz^{-1}}{1 + az^{-1}}, \quad \theta = \begin{bmatrix} a \\ b \end{bmatrix} \qquad (9)$$

After some algebra one obtains the following equivalent input/output description of (8):

$$A_\theta y - B_\theta u = B_\theta e \qquad (10)$$

where $e(t)$ is an uncertain sequence satisfying

$$|e(t)| \leq \frac{\delta}{k}|u(t)|, \quad \forall t = 1, \ldots, N \tag{11}$$

Since (k, δ, ρ) are known and $u(t)$ is measured, the upper bound on the error sequence is known at each time instant. Combining the above expressions with prior information $\theta \in \Theta_0$, we obtain the parameter set estimate

$$\Theta_0 \cap \Theta_N$$

where Θ_N consist of those θ satisfying,

$$|y(t) + ay(t-1) - bu(t-1)| \leq \frac{\delta}{k}|bu(t-1)|, \tag{12}$$

for all $t = 1, \ldots, N$.

Example 2: Output Nonlinearity

In the above example, the nonlinearity was on the input. Now consider the case where the nonlinearity is on the output (see Figure 5) where

$$y = f(\bar{y}), \quad \bar{y} = G_\theta u \tag{13}$$

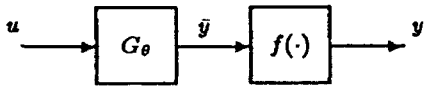

Figure 5: Input nonlinearity.

Proceding as before we now have,

$$A_\theta y - B_\theta u = A_\theta e \tag{14}$$

where now $e(t)$ is an uncertain sequence satisfying

$$|e(t)| \leq \frac{\delta}{k-\delta}|y(t)|, \quad \forall t = 1, \ldots, N \tag{15}$$

In this case the set estimate Θ_N consists of those θ satisfying,

$$|y(t)+ay(t-1)-bu(t-1)| \leq \frac{\delta}{k-\delta}\left(|y(t)| + |ay(t-1)|\right), \tag{16}$$

for all $t = 1, \ldots, N$.

Example 3: Mechanical System

Consider the mechanical configuration depicted in Figure 6.

This system represents the case where tortional actuation is applied to a load through a flexible gear-train. The gearing is shown to occur at the end of the flexible member, although other combinations are certainly possible.

Neglecting any electronic dynamics, and assuming that the flexible rod is both uniform and damped, the motion of the

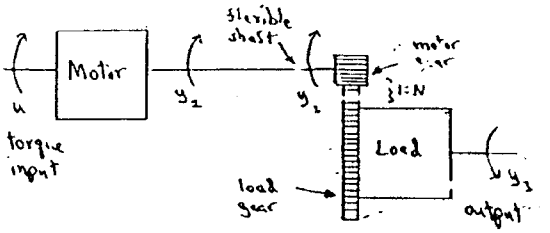

Figure 6: A flexible rotating system with backlash in the gear-train.

rigid body and first tortional "mode" for small angular deflections can be approximated by the system of differential equations,

$$\begin{aligned} J_M\ddot{y}_1 &= u + D(\dot{y}_2 - \dot{y}_1) + K(y_2 - y_1) \\ J_G\ddot{y}_2 &= -\bar{u} - D(\dot{y}_2 - \dot{y}_1) - K(y_2 - y_1) \\ J_L\ddot{y}_3 &= N\bar{u} \\ \bar{y} &= y_2 - Ny_3 \\ \bar{u} &= f(\bar{y}) \end{aligned}$$

where u denotes the input applied torque, (y_1, y_2, y_3) are angular deflections as indicated in the figure, $\bar{y}$ is the relative gear angle, and $f(\cdot)$ is a memoryless nonlinearity arising from backlash in the gear train. The constants are defined as follows: J_M, J_G, and J_L are the motor, motor gear, and load inertias, respectively, N is the gear ratio which is greator than one, and D, K are the damping and stiffness, respectively, of the elastic rod. The backlash nonlinearity $f(\cdot)$ has the typical shape as shown in Figure 7.

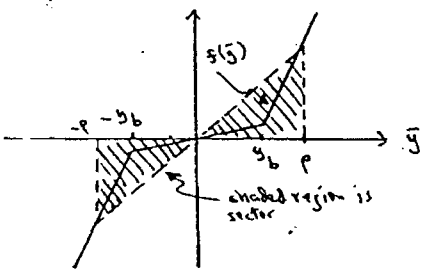

Figure 7: A typical backlash function.

The break-point parameter y_b relates to gear teeth spacing and the slopes in the two regions relate to gear teeth shapes. Typically for $|z| > y_b$ the slope is very large whereas for $|z| < y_b$ the slope is very small. It is clear that for some positive constants (k, δ, ρ) that $f(\cdot)$ satisfies the sector condition (7).

To illustrate how to compute a set estimate for the parameters of the mechanical system, suppose that the measured variables are

$$y = \begin{bmatrix} y_1 \\ y_3 \end{bmatrix}$$

and that (K, D) are uncertain parameters, i.e. ,

$$\theta = \begin{bmatrix} K \\ D \end{bmatrix}$$

Observe that the input to the nonlinear function is

$$\bar{u} = \frac{J_L}{N}\bar{y}_3$$

One approach to describing a model set for this type of system is to approximate either the input or output to the nonlinear function. This would be like the ideal situations in the preceding two example systems. In this case, since y_3 is available as a measurement, the acceleration $\bar{y}_3$ can be approximated by high pass filtering the measured output y_3. For example, let

$$\widehat{\bar{y}}_3 = Fy_3, \quad F(s) = \left(\frac{s}{s\tau + 1}\right)^2$$

where $1/\tau$ is sufficiently large so as to capture the dominant harmonics in the acceleration. Then,

$$\bar{u} \approx \hat{u} = \frac{J_L}{N}\widehat{\bar{y}}_3$$

With this approximation the situation is very similar to the example where the nonlinearity is on the output. However, there is one difference: here the input to the nonlinear function also contains a term do to $\bar{u}$, which can also be approximated by $\hat{u}$. Thus, the appropriate model can be described by the *feedback system* shown in Figure 8.

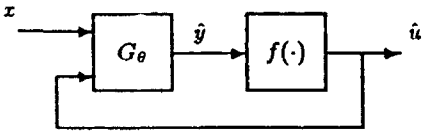

Figure 8: Feedback nonlinearity.

The system is described by,

$$\hat{u} = f(\hat{y}), \quad \hat{y} = G_\theta \begin{bmatrix} x \\ \hat{u} \end{bmatrix}, \quad x = \begin{bmatrix} y \\ u \end{bmatrix} \tag{17}$$

After some algebra, we obtain,

$$\bar{y} \approx \hat{\bar{y}} = -\frac{J_M\hat{u} + J_G u}{A_\theta} + y_1 - Ny_3$$

where

$$A_\theta(s) = J_M J_G s^2 + (J_M + J_G)(Ds + K)$$

The procedure described in Example 1 can now be applied to obtain a set estimate which will contain the true parameters. Of course the precise conditions under which the true parameters are in the set estimate involve various approximations. In particular, due consideration must be given to approximating $\bar{u}$ by $\hat{u}$.

2.3 Standard Model Structure

Even though the three example systems are fairly general, it is also important to point out that they do not exhaust all the myriad possibilities. A very general model format, or template, is characterized in Figure 9.

This model form is discussed in detail in [15]. Here we make the following assumptions:

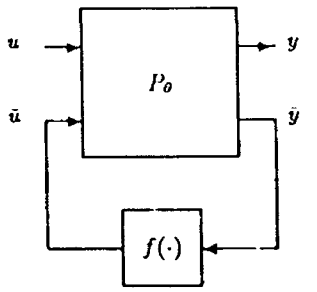

Figure 9: Standard model

1. P_θ is a transfer matrix which depends on a parameter $\theta \in \mathbb{R}^p$ and which has the block structure:

$$\begin{bmatrix} y \\ \bar{y} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} u \\ \bar{u} \end{bmatrix} \tag{18}$$

2. $f(\cdot)$ is a scalar memoryless nonlinearity in the sector,

$$\alpha z \le f(z) \le \beta z, \quad \forall |z| \le \rho$$

where $0 < \alpha < \beta$.

3. The measured data set is

$$\{y(t), u(t) \ : \ t = 1, \ldots, N\}$$

The standard form allows for scalar memoryless sector bounded nonlinearties, but the measured signals (y, u) can be vectors. Disturbances as well as nonparametric dynamic uncertainties can also be included by replacing the "feedback" with a more complicated system and by adding another input.

3 Robust Linear Control Design

As an illustrative example, consider the uncertain nonlinear plant with a linear feedback control,

$$y = d + f(\bar{y}), \quad \bar{y} = G_\theta u, \quad u = -Ky \tag{19}$$

where G_θ and K are linear-time-invariant systems, K is the linear feedback controller, $f(\cdot)$ is a memoryless nonlinearity, y is the measured output to be controlled, and d is a disturbance as seen at the output. The control objective is to attenuate the effect of the disturbance at the output despite the uncertainties in the system model. Specifically, the system uncertainties are as follows:

- the nonlinear function $f(\cdot)$ is in the sector,

$$|f(\bar{y}) - \bar{y}| \le \delta|\bar{y}|, \quad \forall |\bar{y}| \le \rho$$

- the parameters in the linear-time-invariant system G_θ are in the set $\hat{\Theta}$.

Observe that these uncertainty sets can arise from a combination of set estimation and/or prior information. From the control design viewpoint the source of the uncertainty is not relavent.

To analyze this system we make the following convenient defintions:

$$\begin{aligned}\Delta(\bar{y}) &= f(\bar{y}) - \bar{y} \\ S_\theta &= (1 + G_\theta K)^{-1} \\ T_\theta &= G_\theta K(1 + G_\theta K)^{-1} = 1 - S_\theta\end{aligned}$$

Observe that $\Delta(\cdot)$ satisfies the sector condition

$$|\Delta(\bar{y})| \leq \delta|\bar{y}|, \quad \forall|\bar{y}| \leq \rho$$

The transfer functions (S_θ, T_θ) are the closed-loop transfer functions from disturbance d to output y and control u, respectively, if the nonlinear function $f(\cdot)$ is replaced by the identity operator, which in this case is the "nominal" nonlinearity. The nonlinear feedback system is then equivalently expressed as:

$$\begin{aligned}y &= S_\theta(d + \epsilon) \\ \epsilon &= \Delta(\bar{y}) \\ \bar{y} &= -T_\theta(d + \epsilon)\end{aligned}$$

Now, let $T_\theta(t)$ denote the impulse response of $T_\theta(s)$, and suppose that there are constants $M \geq 1$, $\alpha > 0$, and $r > 0$, independent of θ, such that for all $t \geq 0$,

$$\begin{aligned}|T_\theta(t)| &\leq Me^{-\alpha t} \\ |(T_\theta d)(t)| &\leq r\end{aligned}$$

Application of the Bellman inequality [11] yields:

$$|\epsilon(t)| \leq \frac{\delta r}{1 - \delta M/a}$$

provided that

$$\begin{aligned}\delta &< \frac{a}{M} \\ r &< (1 - \delta M/a)\,\rho\end{aligned}$$

The above inequalities bound $\epsilon(t)$, which appears as an additional disturbance. Thus, the ideal closed-loop transfer functions (S_θ, T_θ) must be shaped to make $\epsilon(t)$ small. In addition, the linear controller K has other goals *e.g.* ,to robustly stabilize the linear-time-invariant model set $\{G_\theta : \theta \in \hat{\Theta}\}$.

4 Concluding Remarks

A separation principal between model set estimation and robust control design allows for a more comprehensible approach to adaptive control design. This approach differs from its predecessors in that model uncertainty is incorporated in the *synthesis* phase of the design rather than in the *anlysis* phase.

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A Family of Norms For System Identification Problems

Mohammad-Ali Massoumnia*†
Robert L. Kosut‡

Abstract

In this paper we introduce a family of norms that may prove useful in system identification problems. The important property of the new norm is that for a given sequence its value in the limit will converge to the supremum over all frequencies of the spectrum of the sequence. Using this property, a procedure is outlined to approximately minimize the weighted $\mathcal{L}_\infty$ norm of the frequency response estimation error.

1 Introduction

The parametric approach to system identification is based on selecting an appropriate model structure and a search for the parameters of the model that best describes the data. Usually, the best model within the model set is characterized as the one that minimizes a selected norm of the prediction errors. By far the most popular norm is the sum of the square of the prediction errors– the quadratic norm. In this paper we introduce a new family of norms that seem to be useful in system identification problems. The new norms have an interesting interpretation in the frequency domain and include the usual quadratic norm as a special case. The important property of the new norm is that in the limit its minimization is equivalent to minimizing the supremum over all frequencies of the spectrum of the prediction error, or equivalently minimizing its $\mathcal{L}_\infty$ norm.

2 Definitions and Preliminaries

Let us assume we are given a scalar bounded sequence $\{e_i,\ i = 1, \ldots, N\}$ which in our application represents the prediction errors computed from the observed data and a guessed model parameter vector θ. Based on this sequence, form the $(N + M - 1) \times M$ matrix

$$E_{NM} = \frac{1}{\sqrt{N}} \begin{bmatrix} e_1 & 0 & \cdots & 0 \\ e_2 & e_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_M & e_{M-1} & \ddots & e_1 \\ \vdots & \vdots & \vdots & \vdots \\ e_N & e_{N-1} & \cdots & e_{N-M+1} \\ 0 & e_N & \cdots & e_{N-M+2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_N \end{bmatrix} \tag{1}$$

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†Research Scientist, Integrated Systems Inc., 3260 Jay Street, Santa Clara, CA 95054

‡Manager, Basic research group, Integrated Systems Inc.

where we assume $1 \leq M \leq N$. Note that E_{NM} is constant along the diagonals, and for $M = 1$, E_{N1} is a column vector with $e_i/\sqrt{N}$ as its elements. To simplify the notation we denote this vector by E_N. Moreover, the matrix E_{NM} is completely specified when $E_N(= E_{N1})$ and the value of M are given.

It is simple to see that the matrix $E_{NM}^T E_{NM}$ is symmetric, at least positive semi-definite, and Toeplitz. The elements of this matrix are estimates of the autocorrelation function of the sequence e_i. More explicitly, define the sequence a_i $(i = 0, \ldots, M-1)$ in terms of e_i as follows:

$$a_i = \frac{1}{N} \sum_{j=1}^{N-i} e_j e_{j+i} \tag{2}$$

Then a simple computation shows:

$$E_{NM}^T E_{NM} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{M-1} \\ a_1 & a_0 & \ddots & a_{M-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M-1} & a_{M-2} & \cdots & a_0 \end{bmatrix} \tag{3}$$

Using these definitions, we define the new norm as the maximum eigenvalue of $E_{NM}^T E_{NM}$,

$$V_M(E_N) = \overline{\lambda}\left(E_{NM}^T E_{NM}\right) = \overline{\sigma}^2(E_{NM}) \tag{4}$$

where $\overline{\lambda}(F)$ denotes the maximum eigenvalue of F and $\overline{\sigma}(F)$ denotes the maximum singular value of F. For simplicity, we usually delete the argument of V_M and assume it is understood to be a function of E_N which is itself formed from the prediction errors e_i. Note that V_M defined in (4) is not mathematically a norm on $\mathcal{R}^N$; however $\sqrt{V_M(E_N)}$ *is a valid norm for* E_N, and only to simplify the presentation we refer to V_M as a norm.

Also, for $M = 1$, V_M is identified with the usual quadratic norm. From another point of view, V_1 only includes an estimate of the autocorrelation function of the prediction error for zero shift, a_0. Moreover, V_M is nicely bounded by V_1 as follows:

$$||E_N||_2^2 = V_1(E_N) \leq V_M(E_N) \leq M V_1(E_N) = M||E_N||_2^2 \tag{5}$$

To illustarte some of the properties of V_M for $M > 1$, assume $M = 2$. The maximum eigenvalue of $E_{N2}^T E_{N2}$ is simple to compute and is given by

$$V_2 = a_0 + |a_1| \tag{6}$$

In this case, not only the sum of square of prediction errors is included in the performance measure, but this norm also includes an estimate of the autocorrelation function of the prediction error at the first time shift. Therefore, minimizing V_2 will force $|a_1|$ to small values. *This is a first attempt to whitening the prediction error in addition to minimizing its variance* .

Note that the whiteness of the prediction error is an important factor in validating a computed model [5]. However, this desirable property of the prediction error is not reflected in any form in the usual quadratic norm. But V_M not only is a function of the variance of the prediction error but it also is a function of the values of the autocorrelation of the prediction error for time shifts up to $M-1$, and by increasing M more and more of the temporal behavior of this autocorrelation affects V_M.

3 Frequency Domain Properties

Now we discuss the frequency domain interpretation of the new norm. First assume the limit of a_i defined in (2) as N goes to infinity exists:

$$\lim_{N\to\infty} a_i = \overline{a}_i \tag{7}$$

If, in addition, $\overline{a}_i$ is in l_1, then the spectrum of the prediction error is

$$S_{ee}(\omega) = \sum_{k=-\infty}^{\infty} \overline{a}_k e^{-j\omega k} \tag{8}$$

where we set $\overline{a}_{-k} = \overline{a}_k$ because we are dealing with a real sequence. It is shown in [3] that the following are true

$$\overline{a}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{ee}(\omega)d\omega \tag{9}$$

$$\overline{\lambda}(C_M) \leq \sup_{|\omega|\leq\pi} S_{ee}(\omega) = \lim_{M\to\infty} \overline{\lambda}(C_M) \tag{10}$$

$$\underline{\lambda}(C_M) \geq \inf_{|\omega|\leq\pi} S_{ee}(\omega) = \lim_{M\to\infty} \underline{\lambda}(C_M) \tag{11}$$

where $\underline{\lambda}(F)$ and $\overline{\lambda}(F)$ denote respectively the smallest and the largest eigenvalue of F, and the Toeplitz matrix C_M is defined as follows:

$$C_M = \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \cdots & \overline{a}_{M-1} \\ \overline{a}_1 & \overline{a}_0 & \ddots & \overline{a}_{M-2} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{M-1} & \overline{a}_{M-2} & \cdots & \overline{a}_0 \end{bmatrix} \tag{12}$$

To explore the convergence property of (10) as M goes to infinity, let us consider an exponentially correlated sequence e_i with autocorrelation function given by

$$\overline{a}_k = r^{|k|}, \quad 0 < r < 1 \tag{13}$$

The spectrum of e_i is simple to compute and is given by

$$S_{ee}(\omega) = \frac{1-r^2}{1-2r\cos\omega + r^2} \tag{14}$$

Let us denote the supremum of $S_{ee}(\omega)$ by S. In Figure 1 the values of $100(S-\overline{\lambda}(C_M))/S$ are shown as a function of M for values of r from 0.1 to 0.9. As can be seen, for small values of r (slightly correlated sequences) the convergence is rather fast. However, as r gets closer to one the number M for achieving a preset accuracy increases considerably. This figure is very useful in selecting an appropriate value for M when a bound for the spectral content of the prediction error is known. Moreover, explicit computation shows that for $\overline{a}_k$ given in (13) the convergence of $\underline{\lambda}(C_M)$ to $\inf_\omega S_{ee}(\omega)$ is considerably faster than those observed in Figure 1.

Theorem 1 *The following limits hold*

$$\lim_{N\to\infty} E_N^T E_N = \frac{1}{2\pi}\int_{\pi}^{\pi} S_{ee}(\omega)d\omega \tag{15}$$

$$\lim_{M\to\infty}\left(\lim_{N\to\infty} \overline{\sigma}^2(E_{NM})\right) = \sup_{|\omega|\leq\pi} S_{ee}(\omega) \tag{16}$$

$$\lim_{M\to\infty}\left(\lim_{N\to\infty} \underline{\sigma}^2(E_{NM})\right) = \inf_{|\omega|\leq\pi} S_{ee}(\omega) \tag{17}$$

where we assume that N goes to infinity faster than M.

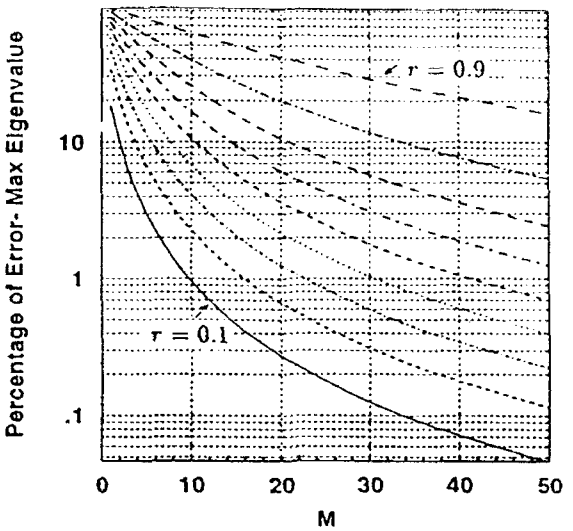

Figure 1: Convergence of $\overline{\lambda}(C_M)$ to $\sup_\omega S_{ee}$

Proof: Relation (15) follows from the definition of $\overline{a}_0$ and (9). Moreover, by definition of C_M we have

$$\lim_{N\to\infty} E_{NM}^T E_{NM} = C_M \tag{18}$$

Substituting this in (10) and (11) and noting that the eigenvalues of a matrix are continous functions of the elements of the matrix the other results follow immediately.

□

In identification problems we estimate the model parameters θ by minimizing $V_M(E_N(\theta))$. The notation $E_N(\theta)$ emphasizes the fact that the prediction error is a function of θ and the minimization is carried over elements of θ. Relation (16) is very illuminating in this respect and shows that by minimizing V_M as M approaches infinity, the supremum over all frequencies of the spectrum of prediction error is minimized. Because of this property, we refer to the identification problem using the new norm as the $\mathcal{L}_\infty$ *identification problem*. In contrast, by minimizing the usual quadratic norm, the integral of the spectrum of prediction error over all frequencies is minimized [5], and this can be referred to as $\mathcal{L}_2$ *identification problem* (see (15)).

As an aside, using (16) and (17), it is clear that *the condition number of E_{NM} is a good indication of the whiteness of the sequence E_N*. When this condition number is close to 1, the spectral density function is close to being constant over all frequencies and the sequence is close to being uncorrelated. Large values of the condition number indicate that the sequence is correlated and the maximum and minimum value of the spectral density are far apart.

Now we explore the usefullness of the new norm in identification problems and relate the $\mathcal{L}_\infty$ norm of the spectrum of the prediction error to $\mathcal{L}_\infty$ norm of the transfer function estimation error. Following the procedure used in [5], let us assume the true system output is generated by

$$y_t = G_0(q)u_t + v_t \tag{19}$$

where the additive noise v_t has the spectrum

$$S_{vv}(\omega) = \lambda_0 |H_0(e^{j\omega})|^2 \tag{20}$$

with $H_0(\infty) = 1$. Also assume the suggested model for the system has the form

$$y_t = G(q,\theta)u_t + H(q,\theta)e_t \tag{21}$$

where θ is the vector of unknown parameters. It is simple to show that the spectrum of the prediction error in this case is given by [5]:

$$S_{ee}(\omega,\theta) = \frac{|\tilde{G}(e^{j\omega},\theta)|^2 S_{uu}(\omega) + S_{vv}(\omega)}{|H(e^{j\omega},\theta)|^2} \tag{22}$$

where $\tilde{G} = G - G_0$ is the error in estimating the transfer function.

Unfortunately, the term $S_{vv}/|H|^2$ in (22) which depends on the parameter θ prevents us from directly relating the minimization of S_{ee} to the minimization of $|\tilde{G}|$. To circumvent this difficulty, we can first use a high order ARX model

$$A(q)y_t = B(q)u_t + e_t \tag{23}$$

to approximate $H_0(e^{j\omega})$ by $1/A(e^{j\omega})$, and filter both u_t and y_t by $A(q)$. Let us denote the filtered input and output by u_t^f and y_t^f respectively. Next use the following output error model to estimate the model parameters θ

$$y_t^f = G(q,\theta)u_t^f + e_t \tag{24}$$

Now using (22) we have

$$S_{ee}(\omega,\theta) = |\tilde{G}(e^{j\omega},\theta)|^2 |A(e^{j\omega})|^2 S_{uu}(\omega) + |A(e^{j\omega})|^2 S_{vv}(\omega) \tag{25}$$

If $1/|A|$ is a good approximation to $|H_0|$, then the last term in (25) is a constant equal to λ_0, and we can write

$$|\tilde{G}(e^{j\omega},\theta)|^2 |A(e^{j\omega})|^2 S_{uu}(\omega) \approx S_{ee}(\omega,\theta) - \lambda_0 \tag{26}$$

Using (26), it is clear that *minimizing the supremum of S_{ee} in this case will directly lead to the minimization of the weighted $\mathcal{L}_\infty$ norm of $\tilde{G}$*. Note that as is expected, the weighting $|A|^2 S_{uu}$ ($\approx S_{uu}/|H_0|^2$) puts more emphasis on the frequency ranges where the signal to noise spectral ratio is large. Also, by repeating the experiment with a different input (changing S_{uu}), we have the flexibility of changing this weighting factor.

However, the approach we have outlined has a major draw back because it relies on using the output error form in (24). The norm of the prediction error in this case is not necessarily a convex function of the model parameters, and this may lead to a complicated minimization problem.

Note that after minimizing V_M (for sufficiently large value of M), we can compute a good estimate for the supremum over all frequencies of the left hand side of (26). Since the supremum over all frequencies of the first term on the right hand side of (26) can be approximated by the minimum value of V_M, and the value of λ_0 (variance of the noise) can be approximated when we are computing the ARX structure in (23). This gives a bound for the $\mathcal{L}_\infty$ norm of the modeling error.

4 Convergence and Convexity

The norm introduced in (4) has some interesting properties that we shall discuss next. Let us fix M, and assume we are given a model structure and identify the parameter vector θ of this model by minimizing $V_P(E_N(\theta))$ where P is a positive integer less than M ($P < M$). Let us assume this optimization problem has a unique global minimum that we will denote by θ^P, and denote the prediction error sequence resulting from this choice of the parameter vector by $E_N^P = E_N(\theta^P)$. Similarly define θ^M and $E_N^M = E_N(\theta^M)$.

Theorem 2 *The following series of inequalities hold:*

$$V_P(E_N^P) \leq V_P(E_N^M) \leq V_M(E_N^M) \leq V_M(E_N^P) \tag{27}$$

Proof: Beginning from the left hand side, the first inequality follows from the fact that the elements of E_N^P are generated from the model parameters that minimize V_P. The second inequality follows from the interlacing property of the eigenvalues of a symmetric matrix [2]. Note that $(E_{NP}^M)^T E_{NP}^M$ is the first $P \times P$ principle minor of $(E_{NM}^M)^T E_{NM}^M$ where E_{NP}^M and E_{NM}^M are defined in terms of E_N^M using (1). The third inequality follows from the fact that E_N^M is formed from model parameters that minimize V_M.

□

The relation given in (27) is specially usefull if we set $P = 1$. Then $V_1(E_N^1)$ is the minimum value of the usual least squares performance measure. Also in this case we can add another important inequality to the set given in (27).

Corollary 1 *The following series of inequalities hold*

$$\begin{aligned} V_1(E_N^1) \leq V_1(E_N^M) \leq V_M(E_N^M) \leq V_M(E_N^1) \leq \\ V_1(E_N^1) + (M-1)\max(|a_1^1|, \ldots, |a_{M-1}^1|) \end{aligned} \tag{28}$$

where a_i^1 are computed from the elements of E_N^1 using the relation given in (2).

Proof: The first three inequalities follow by setting $P = 1$ in (27). Moreover, because $Q = (E_{NM}^1)^T E_{NM}^1$ is Toeplitz with a_0^1 on its main diagonal, each eigenvalue of Q denoted by λ satisfies the following inequality

$$|\lambda - a_0^1| \leq (M-1)\max(|a_1^1|, \ldots, |a_{M-1}^1|) \tag{29}$$

This follows from Gershgorin's circle theorem [2] and hence the last inequality in (28) holds. Note that the a_i^1 in (28) are estimates of the autocorrelation function of the prediction error computed from parameters that are obtained by minimizing the quadratic norm.

□

Now let us assume that for a particular problem $V_1(E_N(\theta))$ and $V_M(E_N(\theta))$ both have unique global minimum that are denoted by θ^1 and θ^M respectively. Moreover, let us assume that in this problem, the last term in (28) goes to zero as the number of data points increases. In other words assume for a fixed M we have

$$\lim_{N \to \infty} \max(|a_1^1|, \ldots, |a_{M-1}^1|) = 0 \tag{30}$$

Then using (28), it is clear that

$$\lim_{N \to \infty} V_1(E_N(\theta^1)) = \lim_{N \to \infty} V_1(E_N(\theta^M)) \tag{31}$$

Now using the assumption on the uniqueness of the global minimum of V_1, it is clear that in the limit θ^1 and θ^M will be identical. Put it more loosely, if the prediction error for the quadratic norm minimization is white, then the parameters obtained by minimizing the new norm will be identical to those obtained by minimizing the usual quadratic norm.

To guarantee that each V_1 and V_M have global minima only, let us choose an ARX model for the structure of the system. In this case it is well known that the scaled prediction error can be written as

$$E_N = \frac{1}{\sqrt{N}}(Y - \Phi\theta) \tag{32}$$

where Φ is the matrix of regression vectors and Y is the vector of output values [5]. In this case, $V_1(E_N(\theta))$ is a convex function of θ, and the minimization problem has only global minima. Moreover, if Φ is full column rank, then the minimum is unique.

Next we show that *for an ARX model structure, V_M is also a convex function of the parameters.* To see this, note that the matrix E_{NM} can be written as

$$E_{NM} = \sum_{i=1}^{M} T_i E_N w_i^T = \frac{1}{\sqrt{N}} \sum_{i=1}^{M} T_i (Y - \Phi\theta) w_i^T \tag{33}$$

where $w_i \in \mathcal{R}^M$ is the standard basis column vector with 1 in its i-th entry and all other elements zero. Also the $(N + M - 1) \times N$ matrix T_i is defined as follows:

$$T_i = \begin{bmatrix} 0_{(i-1)\times N} \\ I_{N\times N} \\ 0_{(M-i)\times N} \end{bmatrix} \tag{34}$$

Moreover, denote the i-th column of Φ by ϕ_i and the i-th element of $\theta \in \mathcal{R}^L$ by θ_i. Then (33) can be rewritten as

$$\begin{aligned} E_{NM} &= C_0 + \sum_{j=1}^{L} C_j \theta_j \qquad (35) \\ C_0 &= \frac{1}{\sqrt{N}} \sum_{i=1}^{M} T_i Y w_i^T \\ C_j &= \frac{1}{\sqrt{N}} \sum_{i=1}^{M} T_i \phi_j w_i^T \end{aligned}$$

Note that C_j, $j = 0, \ldots, L$, have the same special structure as E_{NM} namely being constant along the diagonals. Now using (35), it is clear that E_{NM} is affine in θ, and $\overline{\sigma}(E_{NM})$ is a convex function of θ. Therefore, V_M being the square of $\overline{\sigma}(E_{NM})$ is also a convex function of θ and the minimization problem has only global minima in this case.

Using these facts, if we use an ARX model structure and if it happens that the resulting prediction errors are white (and consequently the relation (30) holds), then we are guaranteed that the parameter estimate using the new norm will be same as the parameters using the quadratic norm. This is promising because for the sum square norm and ARX structure there are many established properties [5] that readily extend to the new norm.

However, if the prediction error sequence is not white, which will be the case if the 'true' model does not have an ARX structure, then the estimate given by minimizing V_M will usually be different from those obtained from the quadratic norm minimization. Note that the new norm forces the autocorrelation of the prediction error for nonzero shifts to small values (whitens the prediction error) and this proporty may result in a better estimate of the model parameters (compared to qudratic norm for a given model order) when the true model is not actually inside the model set.

As we have shown previously, for an ARX model structure, the matrix E_{NM} is affine in the parameters and we are interested in minimizing the maximum singular value of E_{NM}. This problem is already discussed in the literature [7, 4] and a recent algorithm is proposed in [1]. However, by exploiting the special structure of the matrices C_i defined in (35), it may be possible to increase the efficiency of the algorithm in [1]. Also in our application, the size of the matrices involved is quite large and special attention should be paid to the memory management and algorithmic implementation; otherwise huge amounts of memory will be required to perform the optimization even for modest values of M and N.

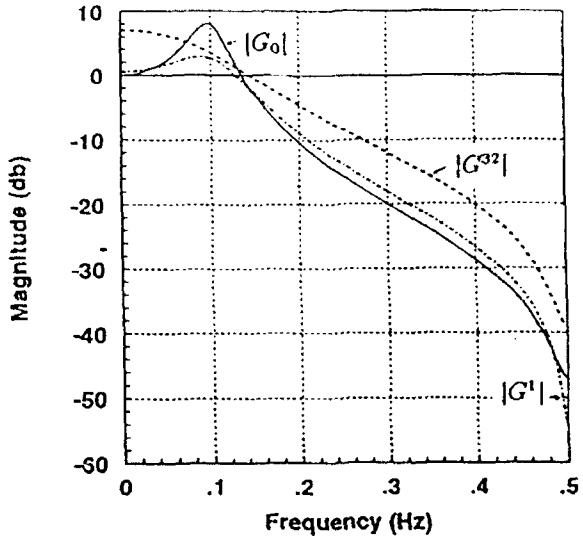

Figure 2: Magnitude Plots

5 Numerical Example

We solved a numerical example to illustrate the properties of the new norm. For performing the minimization, we used the standard OPTIMIZE routine available in MATRIX$_X$ softwrae package [6]. The true model was chosen to be the zero order hold equivalent of a second order lightly damped mode sampled at 1 Hz. The frequency response of the true model is shown in Figure 2 as a solid line. The measured output was assumed to be the sum of the output of the true model and a filtered white guassian pseudo random sequence. The input is a white pseudo random guassian sequence. The signal to noise ratio is chosen to be 5. The number of data points used is 512.

We assumed a second order ARX model for the system and estimated the parameters of the model by minimizing $V_{32}(E_N)$ and $V_1(E_N)$. Note that the true model is in output error [5] form. The resulting estimated transfer functions are denoted by G^{32} and G^1 respectively, and the true transfer function is denoted by G_0 with the magnitude of the frequency responses shown in Figure 2. The magnitude of the errors $G_0 - G^{32}$ and $G_0 - G^1$ are shown in Figure 3. The spectrum and autocorrelation of the prediction errors e_i^{32} and e_i^1 that are obtained from the optimal parameter estimates θ^{32} and θ^1 respectively are shown in Figures 4 and 5. The spectrum is estimated using a Hamming window with a length of 32 points.

For the optimal estimates, the values of the objective functions are as follows:

$$V_{32}(E_N(\theta^{32})) = 0.1926,\ V_1(E_N(\theta^{32})) = 0.1473 \tag{36}$$

$$V_1(E_N(\theta^1)) = 0.0956,\ V_{32}(E_N(\theta^1)) = 0.2781 \tag{37}$$

Note that the values of $V_{32}(E_N(\theta^{32}))$ and $V_{32}(E_N(\theta^1))$ are in close agreement with the maximum of the spectrum of the prediction errors that are shown in Figure 4. Moreover, the values of nonzero shifts of the autocorrelation of e_i^{32} are much smaller than those of e_i^1. In other words, e_i^{32} is close to being white but e_i^1 is clearly correlated. However, the variance of e_i^{32} is considerably larger than that of e_i^1.

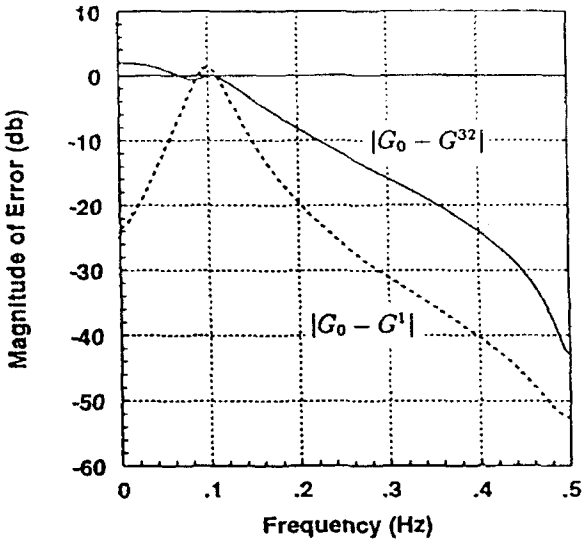

Figure 3: Magnitude of Frequency Response of error

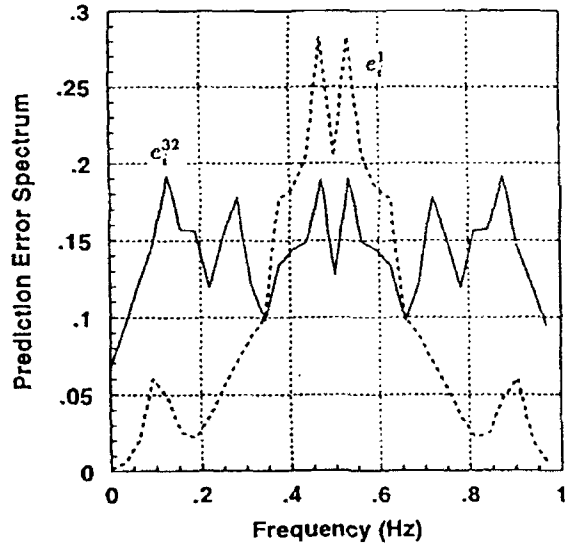

Figure 4: Estimated Spectrums of Prediction Errors

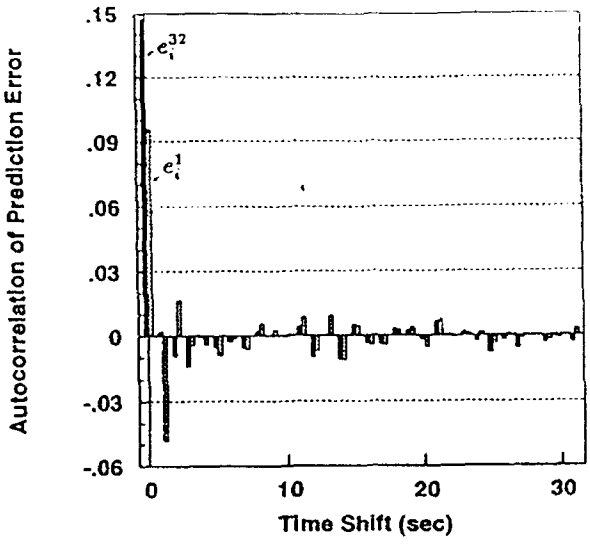

Figure 5: Autocorrelations of Prediction Errors

6 Conclusion

Although we have presented some preliminary results on the properties of the $\mathcal{L}_\infty$ identification problem in this paper, much further work is required to explore the properties of the new norm in details. To perform this task, an efficient implementation of the required minimization algorithm is required so realistic high order models can be estimated and their properties can be compared with those of the least square minimization. As we previously noted, the convexity of the new norm when an ARX model is used is an important property, and hence many techniques of convex optimization can be used for the solution of this problem.

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On Adaptive Robust Control and Control-Relevant System Identification

Wee Sit Lee[1] Brian D. O. Anderson[1] Robert L. Kosut[2] Iven M. Y. Mareels[1]

[1]Department of Systems Engineering
Australian National University
G.P.O. Box 4, Canberra
ACT 2601, Australia

[2]Integrated Systems Inc.
2500 Mission College Blvd.
Santa Clara, CA 95054

Abstract A new approach is given for the design of adaptive robust control in the frequency domain. Starting with an initial model and a robust stabilizing controller, the new (windsurfer) approach allows the bandwidth of the closed-loop system to be increased progressively through an iterative control-relevant system identification and control design procedure. Encouraging results are obtained in the case studies that serve as a benchmark test for the new idea.

1 Introduction

It has long been understood that a key problem in control systems design is to handle the *uncertainties* associated with the plant [12]. Two main techniques for the analysis and design of systems with significant uncertainties are adaptive control [8] and robust control [6, 15].

In the traditional approach to analysis and design of an adaptive control system [8], it is assumed that the unknown plant can be represented by a model in which everything is known except for the values of a finite number of parameters. Once the parameters are estimated (and even during the estimation process), the principle of certainty equivalence is normally invoked to update the controller. Normally the unstructured uncertainties of the model are *ignored* in this approach. Therefore it is not surprising, as pointed out in [18], that these adaptive controllers are often not robust. Further, the extensions of the traditional approach to adaptive control which purportedly cope with unstructured (and other) uncertainties involve conditions which are often hard to apply or to grasp intuitively, see for example [1, 3, 13]. A further problem with the traditional approach is that extreme transient excursions are possible even when global convergence and asymptotic performance are guaranteed [21].

To be more specific, we consider an adaptive control system as shown in figure 1, where G is the unknown transfer function of the plant. The time axis is divided into intervals such that during the i^{th} interval, the control input applied to the plant is obtained from K_i, where K_i is the transfer function of the controller designed using the model G_{i-1} obtained at the end of the $(i-1)^{th}$ time interval.

In an adaptive control problem, the ulterior objective for finding G_i, an estimate of G updated from G_{i-1}, is to redesign a better controller K_{i+1} than K_i, such that certain control objectives are improved. For example if T^d represents the desired complementary sensitivity function, then we may wish to have

$$\left\| \frac{GK_i}{1+GK_i} - T^d \right\|_\infty \leq \left\| \frac{GK_{i-1}}{1+GK_{i-1}} - T^d \right\|_\infty, \quad \forall i.$$

Implicitly, this means we would like to minimize

$$\left\| \frac{GK_i}{1+GK_i} - T^d \right\|_\infty, \quad \forall i.$$

Since G, the transfer function of the plant, is unknown, we could only base our design of K_i on G_{i-1} such that

$$K_i = \arg\min_\gamma \left\| \frac{G_{i-1}\gamma}{1+G_{i-1}\gamma} - T^d \right\|_\infty, \quad \forall i.$$

Note that, as usual, we have invoked the principle of certainty equivalence. However, it is important to realize that

$$\left\| \frac{GK_i}{1+GK_i} - T^d \right\|_\infty$$

is not necessarily small, even though

$$\left\| \frac{G_{i-1}K_i}{1+G_{i-1}K_i} - T^d \right\|_\infty$$

is a minimum. This partly explains why traditional adaptive control systems, which invariably invoked the principle of certainty equivalence, have unsatisfactory robustness property.

In the robust control approach [6, 15], a controller is designed based on a nominal model of the plant with the associated parametric and unstructured model uncertainties explicitly taken into account. Therefore stability robustness is guaranteed and performance robustness is achieved sometimes. The *weakness* of this approach is that it considers only the a priori information on the model, and *neglects* the fact that characteristics of the plant could be learnt while it is being controlled. Therefore, the robust control approach tends to result in a conservative design in terms of performance. It is likely that a posteriori knowledge about the plant could be used to reduce the conservatism in a robust control design.

2 The Windsurfer Approach to Adaptive Control

By considering how humans learn windsurfing, Anderson and Kosut [2] have made the following observations:

1. The human first learns to control over a limited bandwidth, and learning pushes out the bandwidth over which an accurate model of the plant is known.
2. The human first implements a low gain controller, and learning allows the loop to be tightened.

Based on these observations an adaptive robust control design philosophy, *the windsurfer approach*, is proposed in [2]. It recognizes that, at the outset, the plant characteristics can differ greatly from the estimated model at any one time, particularly during the initial learning stage. In the new design approach, a low gain controller will first be implemented; and the control bandwidth will be small. Based on learning a frequency domain description of the

plant in *closed- loop*, with the learning process progressively increasing the bandwidth over which the plant is accurately known, the controller gain can be increased appropriately over an increasing frequency band. For details, refer to [2]. Importantly, in the method suggested, the necessary closed-loop system identification task is simplified into an open-loop system identification problem through the use of coprime fractional representations as discussed in [9, 10].

It was shown recently in [19] that the best model for control design cannot be derived from open-loop experiments alone. The controller to be implemented should be taken into account by the system identification experiments. However, this controller is not yet available, as its determination rests on the results of the system identification to be carried out. Hence, a general solution to the combination of system identification and control design is necessarily iterative. It was also shown in [22] that an iterative approach for model refinement and control robustness enhancement can be developed for a H_2 control problem. Although the emphasis of [19] is on the problem of modeling for control design, its approach is very similar to that of [2]. In the next section, we would like to illustrate the windsurfer approach by considering a model matching problem in the context of adaptive control.

3 Adaptive Model Matching

Let G be the unknown transfer function of the plant, and let T^d represent a desired complementary sensitivity function. We wish to achieve, through iterative system identification and control design, the minimization of the cost function

$$\left\| \frac{GK}{1+GK} - T^d \right\|_\infty ,$$

where K is the transfer function of a controller to be design.

We begin by designing a controller $K_{1,0}$ to stabilize a known initial model G_0, which may be obtained from an open-loop system identification exercise. If $K_{1,0}$ also stabilizes the unknown transfer function G, then we say that $K_{1,0}$ robustly stabilizes G_0. Notice that we use $K_{j,i}$ to denote the j^{th} controller designed using the i^{th} model which has a transfer function G_i. In general, we attach the subscript j,i to a transfer function to denote that it is either spicified or derived on the basis of the i^{th} model for the plant at the j^{th} iteration of control design. Since G_0 may involve significant uncertainties, the resulting controller $K_{1,0}$ may not be able to achieve a small value for

$$\left\| \frac{G_0K_{1,0}}{1+G_0K_{1,0}} - T^d \right\|_\infty$$

while robustly stabilizing G_0. In general, we need to consider how to handle the question of securing robust stabilization of G_i by $K_{j,i}$. This is bound up with the question of selection of T^d. It is in fact to be expected that a sequence of T^d will be selected in such a way that the end control objective can be approached in stages. We shall therefore proceed as follows.

Associated with each of the models G_i, a sequence of controllers $K_{j,i}$ is to be designed such that

$$K_{j,i} = \arg\min_\gamma \left\| \frac{G_i\gamma}{1+G_i\gamma} - T^d_{j,i} \right\|_\infty , \quad \forall j, \tag{3.1}$$

where the sequence of functions $T^d_{j,i}$ is specified with $T^d_{j+1,i}$ normally of wider bandwidth than $T^d_{j,i}$, and with $T^d_{1,i}$ resulting in a controller $K_{1,i}$ that robustly stabilizes G_i. A stage will be reached (say when $j = N$) where the bandwidth of the nominal closed-loop transfer function,

$$\tilde{T}_{N,i} = \frac{G_iK_{N,i}}{1+G_iK_{N,i}}, \tag{3.2}$$

cannot be increased further without causing the effects of model uncertainties in G_i to be too significant. This occurs when the value of

$$\|T_{N,i} - \tilde{T}_{N,i}\|_\infty$$

is no longer small, where

$$T_{N,i} = \frac{GK_{N,i}}{1+GK_{N,i}} \tag{3.3}$$

is the actual closed-loop transfer function of the system.

At this stage it is necessary to improve the accuracy of the model in such a way that is relevant to the control objective. This means that we should try to find an updated model G_{i+1} such that

$$G_{i+1} = \arg\min_\theta \left\| \frac{GK_{N,i}}{1+GK_{N,i}} - \frac{\theta K_{N,i}}{1+\theta K_{N,i}} \right\|_\infty . \tag{3.4}$$

Equation 3.4 would be the formulation of a standard rational function approximation problem, provided that G were known. In the simulation (section 6), we shall take this approach by using a known transfer function for G. This serves as a benchmark test of the windsurfer approach as it corresponds to performing system identification with an infinite number of noiseless measurements. It is a topic of further research to deal with this problem in a realistic system identification setting when only a finite number of (possibly noisy) input-output measurements are available.

Once G_{i+1} is found, we can continue to increase the closed-loop bandwidth by repeating the procedure described for G_i previously. However G_{i+1} should be used instead of G_i, and we specify a new sequence of functions $T^d_{j,i+1}$ with $T^d_{1,i+1} = T^d_{N,i}$. The iterative process is continued until the end control objective is achieved or it is prematurely terminated because of one or more of the following constraints:

1. fundamental performance limitations due to right half plane poles and zeros of the plant and/or models [7].
2. unstable model is obtained. (This is a consequence of our simplified control design method. Appropriate extensions of the control design method [15] allow us to deal with this restriction.)
3. finite control energy.

4 Closed-loop System Identification

We first review a method for closed-loop system identification developed by Hansen [10]. Subsequently, in theorem 4.2, we demonstrate that with appropriate signal filtering, Hansen's method provides a suitable framework to carry out the *control-relevant* system identification formulated in section 3. For the sake of expository simplicity, we shall consider only scalar plants. We begin with the following theorem [20]:

Theorem 4.1 *If $K = \frac{X}{Y}$ is a controller, where X and Y are stable proper transfer functions, and if N and D are stable proper transfer functions that satisfy the Bezout identity*

$$NX + DY = 1,$$

then the set of all plants stabilized by the controller K is precisely the set of elements in

$$\mathcal{G} = \{ \frac{N+RY}{D-RX} : R \text{ is a stable proper transfer function} \}.$$

Consider the feedback system shown in figure 4, where y and u are the measured output and the control input, respectively, e is an unpredictable white disturbance, and r_1 and r_2 are user applied inputs. It is assumed that $K_{j,i}$ is a known stabilizing controller, G is unexactly known and possibly unstable, and, as is standard [14], H is imperfectly known, stable and inversely stable. The system identification problem is to obtain improved estimates of G and H from a finite interval of measured and known data $\{y, u, r_1, r_2 : 0 \le t \le T\}$.

Following Hansen [10], we introduce the stable proper transfer functions $X_{j,i}$, $Y_{j,i}$, $N_{j,i}$, and $D_{j,i}$ which satisfy

$$K_{j,i} = \frac{X_{j,i}}{Y_{j,i}},$$

$$G_i = \frac{N_i}{D_i},$$

and

$$N_i X_{j,i} + D_i Y_{j,i} = 1.$$

The interpretation is that G_i is a known but imperfect model of the plant which is also stabilized by $K_{j,i}$. Applying theorem 4.1 as shown in [10], there exist stable proper transfer functions $R_{i,j}$ and $S_{i,j}$, with $S_{i,j}$ also inversely stable, such that

$$G = \frac{N_i + R_{i,j} Y_{j,i}}{D_i - R_{i,j} X_{j,i}}, \tag{4.1}$$

$$H = \frac{S_{i,j}}{D_i - R_{i,j} X_{j,i}}, \tag{4.2}$$

where $R_{i,j}$ denotes the parametrization of G using the i^{th} model and its associated j^{th} controller $K_{j,i}$.

As a result, system identification of G and H in closed-loop is equivalent to system identification of the stable proper transfer functions $R_{i,j}$ and $S_{i,j}$. Using equations 4.1 and 4.2, we can represent the feedback system as shown in figure 4.

From figure 4, we can write

$$\beta = R_{i,j}\alpha + S_{i,j}e, \tag{4.3}$$

where

$$\alpha = X_{j,i}y + Y_{j,i}u, \tag{4.4}$$

and

$$\beta = D_i y - N_i u. \tag{4.5}$$

However, as

$$u = K_{j,i}(r_i - y) + r_2$$

and

$$K_{j,i} = \frac{X_{j,i}}{Y_{j,i}},$$

equation 4.4 can be re-written as

$$\alpha = X_{j,i}r_1 + Y_{j,i}r_2. \tag{4.6}$$

It is important to observe from equations 4.3, 4.5 and 4.6 that α depends on the applied signals r_1 and r_2 operated on by known stable proper transfer functions $X_{j,i}$ and $Y_{j,i}$ respectively, and β depends on measured signals y and u operated by known stable proper transfer functions D_i and N_i respectively. Moreover, α is independent of the transfer functions G and H and the disturbance e. Hence the system identification of G and H in closed-loop has been recast into the system identification of $R_{i,j}$ and $S_{i,j}$ in open-loop. We shall next state a result which is highly relevant to the system identification step of the windsurfer approach to adaptive control.

Theorem 4.2 *Let the controller $K_{j,i}$ stabilize the plant transfer function G and the model transfer function*

$$G_i = \frac{N_i}{D_i},$$

where N_i and D_i are stable proper transfer functions, and let

$$K_{j,i} = \frac{X_{j,i}}{Y_{j,i}},$$

where $X_{j,i}$ and $Y_{j,i}$ are stable proper transfer functions satisfying the Bezout identity

$$N_i X_{j,i} + D_i Y_{j,i} = 1.$$

Let G_{i+1} be another model of G, also stabilized by $K_{j,i}$ and therefore having a description

$$G_{i+1} = \frac{N_i + r_{i,j} Y_{j,i}}{D_i - r_{i,j} X_{j,i}} \tag{4.7}$$

where $r_{i,j}$ is a stable proper transfer function. Also define the filtered output error

$$\xi = Y_{j,i}(\beta - r_{i,j}\alpha),$$

where, with $r_2 = 0$,

$$\alpha = X_{j,i} r_1,$$

$$\beta = D_i y - N_i u,$$

$$r_1 = \textit{reference signal}$$

$$y = \textit{plant output},$$

$$u = \textit{control input}.$$

Thus ξ is an error arising in the (open-loop) identification of $R_{i,j}$ through an estimate $r_{i,j}$. Then the filtered output error can be expressed as

$$\xi = \left(\frac{GK_{j,i}}{1+GK_{j,i}} - \frac{G_{i+1}K_{j,i}}{1+G_{i+1}K_{j,i}}\right) r_1 + \frac{H}{1+GK_{j,i}} e$$

The proof is not given due to space limitations.

Suppose that the value of

$$\left\| \frac{GK_{j,i}}{1+GK_{j,i}} - \frac{G_i K_{j,i}}{1+G_i K_{j,i}} \right\|_\infty \tag{4.8}$$

has become large. As it was described in section 3, we want a new identification of G via G_{i+1} for which

$$\left\| \frac{GK_{j,i}}{1+GK_{j,i}} - \frac{G_{i+1}K_{j,i}}{1+G_{i+1}K_{j,i}} \right\|_\infty \tag{4.9}$$

is small. We are going to use the $r_{i,j}$ parametrization of G_{i+1}. By substituting equations 4.1 and 4.7 into expression 4.9, and noting that

$$K_{j,i} = \frac{X_{j,i}}{Y_{j,i}},$$

we can, after simplification, conclude that

$$\left\| \frac{GK_{j,i}}{1+GK_{j,i}} - \frac{G_{i+1}K_{j,i}}{1+G_{i+1}K_{j,i}} \right\|_\infty = \| Y_{j,i} X_{j,i} (R_{i,j} - r_{i,j}) \|_\infty \tag{4.10}$$

should be small.

Remarks

- Note that

$$T_{j,i} = \frac{GK_{j,i}}{1+GK_{j,i}}$$

is the actual closed-loop transfer function of the system, and

$$\tilde{T}_{j,i} = \frac{G_i K_{j,i}}{1+G_i K_{j,i}}$$

is the nominal closed-loop transfer function of the system. Therefore, using similar substitutions that resulted in equation 4.10, we can obtain

$$T_{j,i} - \tilde{T}_{j,i} = Y_{j,i} X_{j,i} (R_{i,j} - \tilde{R}_{i,j}). \tag{4.11}$$

However, since

$$\tilde{R}_{j,i} \equiv 0, \quad \forall j, \forall i,$$

we therefore have

$$T_{j,i} - \tilde{T}_{j,i} = Y_{j,i} X_{j,i} R_{i,j}. \tag{4.12}$$

By comparing the argument of the H_∞ norm given in expression 4.8 with the left hand side of equation 4.12, we see immediately that when the value of

$$\left\| \frac{GK_{j,i}}{1+GK_{j,i}} - \frac{G_iK_{j,i}}{1+G_iK_{j,i}} \right\|_\infty$$

has became large; that is, when the closed-loop property of the actual system ($T_{j,i}$) is significantly different from the closed-loop property of the nominal system ($\tilde{T}_{j,i}$), the value of

$$\|Y_{j,i}X_{j,i}R_{i,j}\|_\infty$$

will be large.

- From the signals defined in theorem 4.2, we observed that $R_{i,j}$, the transfer function to be identified, is excited by the signal α, where

$$\alpha = X_{j,i}r_1,$$

and

$$X_{j,i} = \frac{K_{j,i}}{1+G_iK_{j,i}}.$$

Since the nominal closed-loop transfer function of the system is

$$\tilde{T}_{j,i} = \frac{G_iK_{j,i}}{1+G_iK_{j,i}},$$

we can write

$$X_{j,i} = \frac{\tilde{T}_{j,i}}{G_i}.$$

Therefore, $X_{j,i}$ will have large magnitude when we try to push the nominal closed-loop bandwidth beyond the nominal open-loop bandwidth. Since a model usually has its uncertainties become significant for frequencies beyond its bandwidth, from figure 4, we see that if the spectrum of r_1 is white, we automatically get the right weighting for the input to $R_{i,j}$ for the system identification scheme.

- It is shown in theorem 4.2 that the effect of e on ξ is g given by $\frac{He}{1+GK_{j,i}}$. Note that this is the effect of e on y attenuated by the sensitivity function of the actual closed-loop system.

5 Approximate Identification of the $R_{i,j}$ Transfer Function for IMC Controller Design

In section 4, we have shown that the closed-loop system identification of the plant transfer function G can be reformulated into an open-loop system identification of the stable proper transfer function $R_{i,j}$ that parametrized the transfer function G via the equation

$$G = \frac{N_i + R_{i,j}Y_{j,i}}{D_i - R_{i,j}X_{j,i}}.$$

In this and the following sections, we shall, for simplicity, study the case where the plant is stable and has no zeros on the imaginary axis of the s-plane, and where the IMC method [15] is used to design the controller $K_{j,i}$. We shall also assume that all estimates G_i of the plant are stable.

If the model

$$G_i = \frac{N_i}{D_i}$$

is also stable, we can let $N_i = G_i$ and $D_i = 1$ so that

$$G = G_i + \frac{R_{i,j}}{1 - R_{i,j}Q_{j,i}}, \tag{5.1}$$

where $Q_{j,i}$ is a stable proper transfer function that parametrized the controller

$$K_{j,i} = \frac{X_{j,i}}{Y_{j,i}},$$

and

$$Q_{j,i} \stackrel{\text{def}}{=} \frac{K_{j,i}}{1+G_iK_{j,i}}. \tag{5.2}$$

Note that

$$X_{j,i} = Q_{j,i},$$

and

$$Y_{j,i} = 1 - Q_{j,i}G_i$$

Since the parametrization of G by $R_{i,j}$ depends intimately on $Q_{j,i}$, we shall briefly explain how $Q_{j,i}$ is obtained in the design of the controller $K_{j,i}$. We will use the notations n_H and d_H to denote the numerator polynomial and the denominator polynomial of a rational transfer function H.

Given a stable model,

$$G_i = \frac{n_{G_i}}{d_{G_i}},$$

where d_{G_i} has no zeros in the closed right half s-plane, if n_{G_i} has no zeros on the imaginary axis of the s-plane, we can write

$$G_i = \frac{\tilde{n}_{G_i}\prod_i(z_i - s)}{d_{G_i}},$$

where all z_i have positive real parts, and $\tilde{n}_{G_i}$ has no zeros in the closed right half s-plane. By writing G_i as

$$G_i = [G_i]_m[G_i]_a,$$

where

$$[G_i]_m = \frac{\tilde{n}_{G_i}\prod_i(z_i^* + s)}{d_{G_i}}; \quad z_i^* \text{ is the complex-conjugate of } z_i,$$

and

$$[G_i]_a = \frac{\prod_i(z_i - s)}{\prod_i(z_i^* + s)},$$

we have factored G_i as a product of its minimum-phase factor $[G_i]_m$, and the associated all-pass factor $[G_i]_a$. We can design a controller, using the internal model control (IMC) approach [15], by setting

$$Q_{j,i} = [G_i]_m^{-1}F_{j,i}, \tag{5.3}$$

where $F_{j,i}$ is a low pass filter of the form

$$F_{j,i} = \left(\frac{\lambda_{j,i}}{s+\lambda_{j,i}}\right)^n,$$

with n chosen large enough so that $Q_{j,i}$ is proper, and $\lambda_{j,i}$ selected (possibly on-line) small enough so that $K_{j,i}$ robustly stabilizes G_i.

In the ideal situation where $G_i = G$ is stable and minimum-phase, it follows that the nominal and the actual closed-loop transfer functions of the system are equal and are given by the transfer function $F_{j,i}$. Therefore $\lambda_{j,i}$ is both the nominal and actual closed-loop system bandwidth with a $-3n\,dB$ attenuation. In general, $G_i \neq G$ and $\lambda_{j,i}$ serves only as an approximate bandwidth of the actual closed-loop system.

With the controller designed using the above procedure, we shall now show that the transfer function to be identified, $R_{i,j}$, is the product of a known stable proper transfer function and an unknown stable strictly-proper transfer function. An analysis of the form of the unknown factor in $R_{i,j}$ indicates how it can be sensibly approximated by a low-order transfer function. We shall first rewrite equation 5.1 as

$$R_{i,j} = \frac{G - G_i}{1 + Q_{j,i}(G - G_i)}. \tag{5.4}$$

Then we can obtain, after substituting equations 5.2 and 5.3 into equation 5.4, and performing some algebraic manipulations,

$$R_{i,j} = \{[G_i]_m d_{F_{j,i}}\}\left\{\frac{d_G n_G - d_G n_{G_i}}{d_{K_{j,i}} d_G + n_{K_{j,i}} n_G}\right\}. \tag{5.5}$$

Note that equation 5.5 can also be written as

$$R_{i,j} = \tilde{R}_{i,j}\hat{R}_{i,j}, \tag{5.6}$$

where

$$\tilde{R}_{i,j} = [G_i]_m d_{F_{j,i}} \tag{5.7}$$

is a known stable proper transfer function, and

$$\hat{R}_{i,j} = \frac{d_{G_i} n_G - d_G n_{G_i}}{d_{K_{j,i}} d_G + n_{K_{j,i}} n_G} \tag{5.8}$$

is an *unknown* stable strictly proper transfer function that depends on the unknown transfer function G. Therefore the problem of identifying $R_{i,j}$ has become one of identifying its unknown factor $\hat{R}_{i,j}$. We shall summarize this important result in the following theorem.

Theorem 5.1 *Consider a plant which has an unknown stable proper transfer function G, and a model with a known stable proper transfer function G_i. If G and G_i have no zeros along the imaginary axis of the s-plane, and*

$$G_i = [G_i]_m [G_i]_a,$$

where $[G_i]_m$ is the minimum-phase factor of G_i, and $[G_i]_a$ is the all-pass factor of G_i, then with

$$Q_{j,i} = [G_i]_m^{-1} F_{j,i}$$

and

$$F_{j,i} = \left(\frac{\lambda_{j,i}}{s + \lambda_{j,i}}\right)^n,$$

where n is chosen such that $Q_{j,i}$ is a stable proper transfer function, the controller

$$K_{j,i} = \frac{Q_{j,i}}{1 - Q_{j,i} G_i}$$

will robustly stabilize G_i for all sufficiently small values of $\lambda_{j,i} \geq 0$. Furthermore, the unknown stable strictly proper transfer function to be identified,

$$R_{i,j} = \frac{G - G_i}{1 + Q_{j,i}(G - G_i)},$$

can be factorized as

$$R_{i,j} = \tilde{R}_{i,j} \hat{R}_{i,j},$$

where $\hat{R}_{i,j}$ is an unknown stable proper transfer function to be identified, and $\tilde{R}_{i,j}$ is a known stable proper transfer function given by

$$\tilde{R}_{i,j} = [G_i]_m d_{F_{j,i}},$$

where $d_{F_{j,i}}$ is the denominator polynomial of the filter $F_{j,i}$.

Remarks

- Note that the factorization of $R_{i,j}$ given in theorem 5.1 is naturally induced by the IMC [15] controller design procedure that we have adopted.
- The poles of $\hat{R}_{i,j}$ are the poles of $T_{j,i}$, the actual closed-loop transfer function of the system.
- It is important to note that $\hat{R}_{i,j} = 0$ if and only if $G = G_i$.
- The order of $\hat{R}_{i,j}$ is constraint by the degree of the polynomial $d_{K_{j,i}} d_G$, which is an unknown.

As we do not know the order of $\hat{R}_{i,j}$ a priori, and since only step response information is available, it is reasonable to employ a low-order transfer function for the approximate identification of $\hat{R}_{i,j}$. Since we are going to identify $\hat{R}_{i,j}$ (actually $R_{i,j}$) and update G_i to G_{i+1} when the step response of the actual closed-loop system exhibits unacceptable oscillations and/or overshoots, we expect $\hat{R}_{i,j}$ to have complex-conjugate poles. Therefore, the lowest possible order that we can assume for the transfer function which serve as an approximation of $\hat{R}_{i,j}$ is two.

It was shown in equation 4.10 that the system identification problem is to find

$$r_{i,j} = \arg\min_{\phi} \| \lambda_{j,i}^n Y_{j,i}(R_{i,j} - \phi) \|_\infty \tag{5.9}$$

If we define

$$r_{i,j} = \tilde{R}_{i,j} \tilde{r}_{i,j}, \tag{5.10}$$

where $\tilde{r}_{i,j}$ is an unknown second-order stable strictly proper transfer function, then by substituting equations 5.3, 5.6, and 5.10 into equation 5.9, we can show that the system identification problem becomes one of finding

$$\tilde{r}_{i,j} = \arg\min_{\phi} \| \lambda_{j,i}^n Y_{j,i}(\tilde{R}_{i,j} - \phi) \|_\infty \tag{5.11}$$

Remark

- Since $Y_{j,i}$ is the nominal sensitivity function of the closed-loop system, we immediately see that the frequency shaping in the identification criterion given by equation 5.11 will force the updated model to have small modelling error in the range of frequencies where the nominal sensitivity function cannot be made small by the controller $K_{j,i}$.
- When updating the model using the equation

  $$G_{i+1} = G_i + \frac{r_{i,j}}{1 - r_{i,j} Q_{j,i}},$$

  the order of the model may increase. To prevent the model order from increasing indefinitely, we use a frequency weighted balanced truncation scheme to reduce the order of G_{i+1}. Specifically, we find

  $$\hat{G}_{i+1} = \arg\min_{\eta} \left\| \frac{G_{i+1} K_{j,i}}{1 + G_{i+1} K_{j,i}} - \frac{\eta K_{j,i}}{1 + \eta K_{j,i}} \right\|_\infty,$$

  where $\hat{G}_{i+1}$ is the reduced order model. If the model order is restricted to m, the controller will be at most of order $2m$ (see controller design equations given in theorem 5.1). In this way the controller complexity will be limited.

6 Simulation Results

We shall present some simulation results of applying the wind-surfer approach to the control of a plant with the transfer function

$$G(s) = \frac{9}{(s+1)(s^2 + 0.06s + 9)}.$$

We first summarize the procedure in the following algorithm:

Step 1:

Set $G_i = G_0$, where G_0 is the transfer function of an initial model of the plant.

Step 2:

Factorize G_i as

$$G_i = [G_i]_m [G_i]_a,$$

where $[G_i]_m$ is the minimum-phase factor of G_i, and $[G_i]_a$ is the associated all-pass factor of G_i.

Step 3:

For $j = 1$, find

$$K_{j,i} = \frac{Q_{j,i}}{1 + Q_{j,i} G_i},$$

with

$$Q_{j,i} = [G_i]_m^{-1} F_{j,i},$$

where the positive integer n and the parameter $\lambda_{j,i}$ in the transfer function

$$F_{j,i} = \left(\frac{\lambda_{j,i}}{s + \lambda_{j,i}}\right)^n$$

are chosen such that $Q_{j,i}$ is a stable proper transfer function, and $K_{j,i}$ robustly stabilizes G, in the sense that the step response of the actual closed-loop system has, at most, little oscillations and/or overshoots. Stop here if such a robust stabilizing controller cannot be found. Also stop here if the robust stabilizing controller results in a closed-loop system which meets the specified bandwidth. Otherwise, proceed to the next step.

Step 4:

Let $j = j + 1$ and set $\lambda_{j,i} = \lambda_{j-1,i} + \epsilon$ for small $\epsilon > 0$, and redesign the controller $K_{j,i}$ using the equations given in Step3. Stop here if the design produces a robust stabilizing controller with the closed-loop system satisfying the specified bandwidth. Otherwise, repeat this step if $K_{j,i}$ robustly stabilizes G_i; else proceed to the next step.

Step 5:

Perform rational function approximation to obtain

$$\hat{r}_{i,j} = \arg\min_{\phi} \|\lambda_{j,i}{}^n Y_{j,i}(\hat{R}_{i,j} - \phi)\|_\infty.$$

Then update the model using the following set of equations:

$$\tilde{R}_{i,j} = [G_i]_m d_{F_{j,i}},$$

$$r_{i,j} = \tilde{R}_{i,j}\hat{r}_{i,j},$$

and

$$G_{i+1} = G_i + \frac{r_{i,j}}{1 - r_{i,j}Q_{j,i}}.$$

Step 6:

If G_{i+1} is stable, find the reduced order model

$$\hat{G}_{i+1} = \arg\min_{\eta} \left\| \frac{G_{i+1}K_{j,i}}{1 + G_{i+1}K_{j,i}} - \frac{\eta K_{j,i}}{1 + \eta K_{j,i}} \right\|_\infty.$$

Otherwise, stop here.

Step 7:

Set $G_i = \hat{G}_{i+1}$ and return to Step 2.

Remarks

- In the algorithm, rational function approximation has to be carried out when $\|T_{N,i} - \tilde{T}_{N,i}\|_\infty$ is no longer small. Broadly speaking, this will correspond to a significant difference between the designed nominal performance (depending on G_i and $K_{N,i}$) and the actual performance (depending on G and $K_{N,i}$). In particular, the observed *step response* may exhibit much more oscillations and/or overshoots than the designed values. This is not of course the same thing as guaranteeing that the H_∞ error above has became large, but neither is it unrelated.

- To be more precise, we define the peak gain of a system, whose transfer function is T, by

$$\|T\|_1 = \sup_{\|w\|_\infty \neq 0} \frac{\|Tw\|_\infty}{\|w\|_\infty}.$$

This is also equal to the *total variation* of the system's unit step response [4] defined as the sum of all consecutive peak-to-valley differences in the unit step response. It can be shown [5] that, if T is a stable strictly proper transfer function,

$$\|T\|_\infty \leq \|T\|_1 \leq 2n\|T\|_\infty,$$

where n is the order of the transfer function T. Now we consider the peak error

$$\|T_{N,i} - \tilde{T}_{N,i}\|_1$$

Since

$$\|T_{N,i} - \tilde{T}_{N,i}\|_1 \geq \|T_{N,i}\|_1 - \|\tilde{T}_{N,i}\|_1,$$

therefore, if the observed step response of $\tilde{T}_{N,i}$ exhibits much more oscillations and/or overshoots than the designed step response of $T_{N,i}$, we would expect

$$\|T_{N,i}\|_1 \gg \|\tilde{T}_{N,i}\|_1,$$

and hence,

$$\|T_{N,i} - \tilde{T}_{N,i}\|_1 \gg \epsilon, \quad \epsilon > 0.$$

Since the peak gain also provides a *loose* lower bound for the H_∞ gain, it is likely that

$$\|T_{N,i} - \tilde{T}_{N,i}\|_\infty$$

becomes large when the observed actual step response exhibits *much more* oscillations and/or overshoots than the desired one.

- This explains why, in the simulation, the models are updated whenever the actual step response exhibits *unacceptable* oscillations and/or overshoots.

The simulation results are presented in figure 5 and figure 6. These figures correspond respectively to the following case studies:

- Case 1: the initial model is $G_0(s) = \frac{1.2}{s+1.2}$,
- Case 2: the initial model is $G_0(s) = \frac{0.8}{s+0.8}$.

We present unit step responses at various steps in the system identification/control design iteration, and frequency responses achieved just before the iteration process is stopped.

In the first case study, see figure 5, the bandwidth of the closed-loop system cannot be increased beyond 10 *rad/sec* because we have stopped the iterative system identification and control design process when an unstable model is obtained. Note that only two model updates, G_1 and G_2, are required in the process, and the results are sufficiently good for most practical purposes.

The results for the second case study are given in figure 6. These results show that the closed-loop bandwidth can easily be pushed to 10 *rad/sec* with very good step responses. Note that in this case, the model has to be updated only once.

Remark

- We must emphasize that in these simulations, instead of performing a system identification using input-output measurements, we actually perform the model approximation

$$\hat{r}_{i,j} = \arg\min_{\phi} \|\lambda_{j,i}{}^n Y_{j,i}(\hat{R}_{i,j} - \phi)\|_\infty,$$

where $\hat{R}_{i,j}$ is obtained from the known G. The reasons for doing this are:

1. Our results, although preliminary, serve as a benchmark in the sense that using the transfer function G corresponds to performing system identification with an infinite number of noiseless measurements.
2. We like to know how serious the problems may be due to employing a low-order approximation for $\hat{r}_{i,j}$. This is important for later system identification studies.
3. We are, at this stage, more concerned with the concept of iterative system identification and control design as applied to adaptive robust control, rather than the details.
4. Efficient algorithms for performing H_∞ system identification are still lacking, and the corresponding theory is still not well understood [11, 16, 17].

7 Discussions and Conclusions

We have reviewed in section 1 the strength and weakness of both the traditional adaptive control and the robust control design methods. These methods should be able to complement each other and there should be natural ways in which they could be blended harmoniously. We proposed that one of the possible ways is by the windsurfer approach, which was first mentioned in [2]. We have shown, by simulation, that by starting with a (crude) initial model of the plant and a (small bandwidth) robustly stabilizing controller, the bandwidth of the closed-loop system can be increased progressively through an iterative control-relevant system identification and control design procedure. We shall highlight the following points which we believe are reasons for the success of the approach:

- **The use of control-relevant frequency weighting in the system identification criterion.**
- **Updating of the model when its effects is no longer small in the closed-loop response. This will ensure that model uncertainties are emphasized in the correct range of frequencies.**
- **The controller designed by using the IMC method always has integral action. Therefore it is insensitive to model uncertainties at low frequencies, provided the gain of the model at low frequencies is of the right sign.**
- **The controller designed by using the IMC method induces a natural factorization in the parametrization of the unknown transfer function of the plant. This enable the system identification problem to be solved effectively.**

In conclusion, we would like to emphasize that only the case of stable plant and model is considered in this preliminary study. We will like to address the following problems in the near future:

- The extension of the method to deal with unstable plant and model.
- Use of orthogonalized exponentials in the system identification procedure such that it becomes a convex optimization problem.
- To prove that the algorithm actually converges in some sense.
- To study other control design methods in the context of the windsurfer approach.

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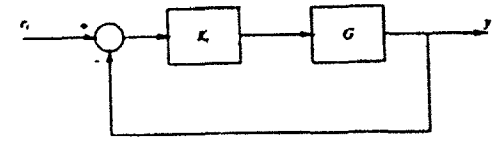

Figure 1: Adaptive control system

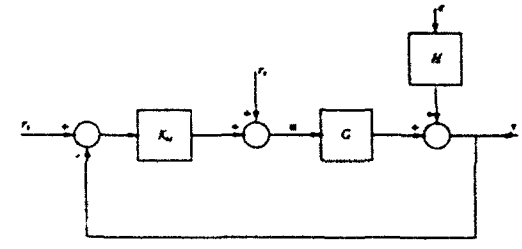

Figure 2: Closed-loop system

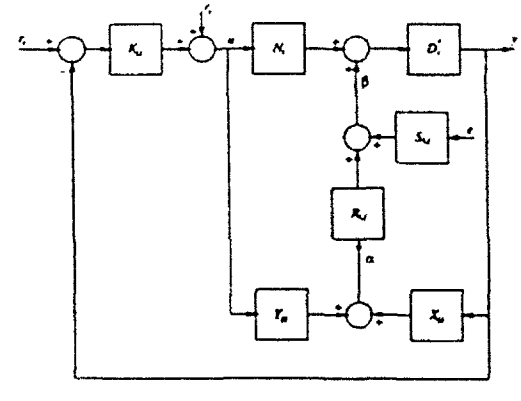

Figure 3: Closed-loop system identification

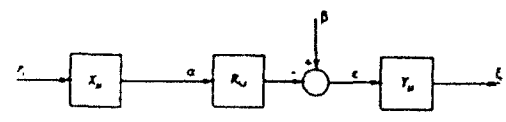

Figure 4: Excitation of R_{ij}

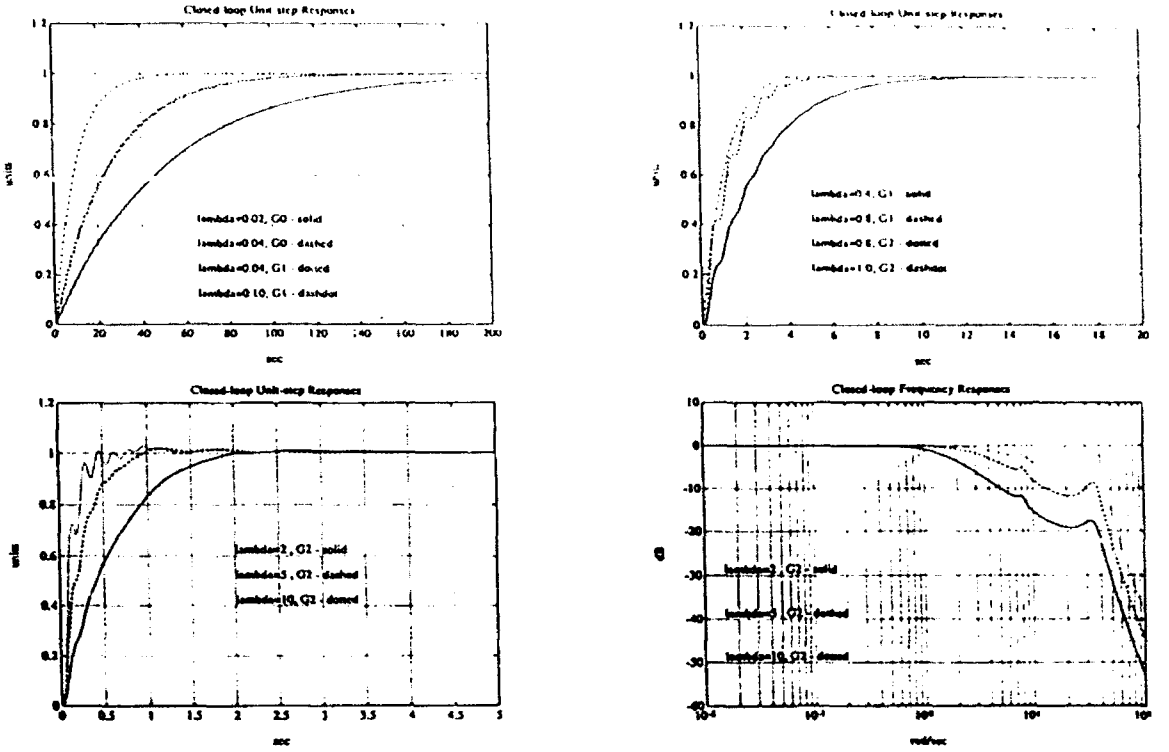

Figure 5: Closed-loop responses with $G_\theta = \frac{1.2}{s+1.2}$

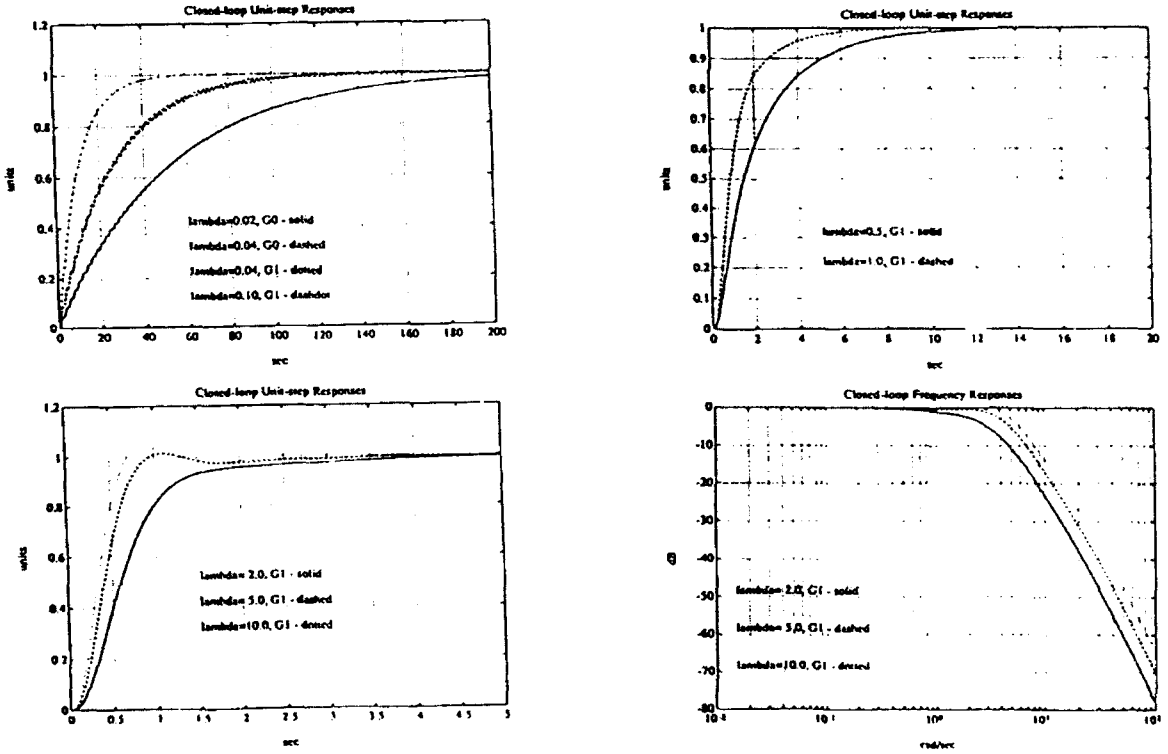

Figure 6: Closed-loop responses with $G_\theta = \frac{0.8}{s+0.8}$

Adaptive Robust Control: On-Line Learning

BRIAN D. O. ANDERSON * ROBERT L. KOSUT †

Abstract A method of on-line adaptation and learning is proposed which makes use of a probing signal whose frequency content is concentrated at the bandwidth of the current controller. As the plant is learned the procedure naturally increases the learning bandwidth.

1 Introduction

It is very easy to *construct* an adaptive system: just connect a controller design rule and a model parameter estimator together. This kind of adaptive control system operates along roughly the following lines. A model for the unknown plant is assumed in which everything is known but the values of a finite number of parameters. These parameters have the property that when they are known, the controller can be defined. It too has a finite number of adjustable parameters, the values of which depend on the plant parameters. By observing the plant input and output, the plant parameters are learned and/or tracked, and the controller parameters are then set according to some design rule. Sometimes it is the controller parameters which are learned directly. Certain choices of controller parametrization lends itself to this approach, others do not.

What is absent in this approach is the recognition that the estimated plant parametric model during the learning phase can be a poor representation of the true plant. This mismatch between the plant and the estimated model can cause poor performance via such phenomena as parameter drifting and bursting. All of this has been reported in the literature and under certain conditions has been analyzed and explained, [1], [2].

In this paper we invoke a different design philosophy than that expressed by the previous reasoning. The new reasoning would have to recognize at the outset that the true plant can differ greatly from the estimated model at any one time, particularly during the initial learning stage.

Nature provides examples of this kind of adaptive control, and it seems that many such examples do not exhibit the traditional operating strategy. In particular, consider how humans learn windsurfing, where the human is the adaptive controller. Several observations can be made: (1) The problem has multiple inputs. (2) The human first learns to control over a limited bandwidth, and learning pushes out the bandwidth. (3) The human first implements a low gain controller; and learning causes the loops to be tightened (this is linked with 2). These observations suggest that one could contemplate an adaptive controller based on learning a frequency domain description of the plant, with the learning process pushing out the bandwidth over which the plant was accurately known. For such a concept to be valid and consistent with point 3 above, it would be necessary to demonstrate, at least for a broad class of plants, that a low gain controller can be contemplated for plants with significant uncertainty at high frequencies, and that reduction in the structured uncertainty progressively allow increase of the controller gain and control over an increasing frequency band; this is essentially a linear systems, as opposed to adaptive systems, exercise.

It would also be desirable to show that when the behaviour of the plant over a certain bandwidth had been learned and certain controller gains implemented, it would be natural to apply a probing signal at the upper limit of this bandwidth (perhaps in handling transients) so that the bandwidth of knowledge of the plant was expanded.

2 Closed-Loop Identification

For the sake of expository simplicity, we shall restrict attention to scalar plants. The following result can be found in one form or another in [9] and the references therein.

Theorem 1 *Suppose that X, Y, N, D are stable transfer functions satisfying*

$$XN + YD = 1 \tag{1}$$

Then:

(i) All controllers C which stabilize the plant $P = N/D$ are in the set of transfer functions ,

$$\left\{ \frac{X + QD}{Y - QN} \ : \ Q \text{ stable} \right\} \tag{2}$$

(ii) All plants P stabilized by the controller $C = X/Y$ are in the set of transfer functions ,

$$\left\{ \frac{N + RY}{D - RX} \ : \ R \text{ stable} \right\} \tag{3}$$

Since all rational transfer functions can be expressed as a ratio of stable transfer functions , it follows that part (i) gives a parametrization of all stabilizing rational controllers of rational plants.

Statement (ii), which follows directly from (i) by interchanging the plant and controller, was developed in [3, 4] for use in closed-loop identification for the problem of experiment design. Similar results are also in [8]. In this paper we also utilize this result, but for a slightly different purpose.

Consider the feedback system,

$$y = Gu + He \tag{4}$$

$$u = K_0(r_1 - y) + r_2 \tag{5}$$

where (y, u) are the measured output and control input, respectively, e is an unpredictable disturbance, and (r_1, r_2) are user applied inputs. It is assumed that K_0 is a stabilizing feedback compensator. This implies some knowlwdge of G, but otherwise G and H are assumed unknown. The *plant* is the pair (G, H) where G is possibly unstable and, as is standard, H and H^{-1} are stable [6]. The identification problem is to obtain estimates of (G, H) from a finite set of measured and known data $\{y, u, r_1, r_2 \ : \ 0 \le t \le T\}$. Following identification, the controller is to be re-designed to improve performance of the closed-loop system.

*Systems Engineering Dept., Australian National University, Canberra, Australia

†Integrated Systems Inc., 3260 Jay St., Santa Clara, CA 95054.
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Stable Plant Let us consider the special case when the plant G is stable. Suppose also that G_0 is stable and that K_0 stabilizes G_0. Then, by Theorem 1, it can be shown that K_0 stabilizes G iff there exists a stable R and stable mini-phase S, such that

$$G = G_0 + \frac{R}{1 - RQ_0}, \quad H = \frac{S}{1 - RQ_0} \tag{6}$$

where

$$Q_0 = \frac{K_0}{1 + G_0 K_0} \tag{7}$$

Again, an interpretation is that K_0 stabilizes all plants in the set

$$\left\{ G_0 + \frac{R}{1 - RQ_0} \ : \ R \text{ stable} \right\} \tag{8}$$

As result, identification of (G, H) in closed-loop is equivalent to identification of the stable open-loop (R, S)-system,

$$\beta = R\alpha + Se \tag{9}$$

where β, α are given by

$$\beta = y - G_0 u \tag{10}$$

$$\alpha = Q_0 r_1 + (1 - Q_0 G_0) r_2 \tag{11}$$

Observe that (α, β) depend on measured and applied signals (y, u, r_1, r_2) operated on by known stable systems (G_0, Q_0).

Example To further motivate identifying the (R, S)-system, consider the following example:

$$G = \frac{9}{(s+1)(s^2 + .06s + 9)}$$

$$G_0 = \frac{1}{s+1}$$

$$Q_0 = \frac{4(s+1)}{(s+2)^2}$$

Figure 1 shows the magnitude of R and $G - G_0$ vs. frequency. These are very close showing that identification of R is close to identification of the model error $G - G_0$.

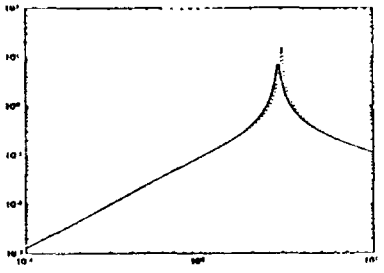

Figure 1: Magnitude plots of R and $G - G_0$ vs. frequency.

Thus, we are led to the following iterative identification algorithm for stable plants in closed-loop. A similar formulation is available for the general case where the plant is possibly unstable.

Initialize: $\hat{G} = G_{00}, \ \hat{Q} = Q_{00} = \dfrac{K_{00}}{1 + G_{00} K_{00}}$

Update $G_0 = \hat{G}, \ Q_0 = \hat{Q}, \ K_0 = \dfrac{Q_0}{1 - Q_0 G_0}$

Identification input: $u = K_0(r_1 - y) + r_2$

R – Update $\hat{R} = \arg\min_R \| y - G_0 u - R(Q_0 r_1 + (1 - Q_0 G_0) r_2) \|$

G – Update $\hat{G} = G_0 + \dfrac{\hat{R}}{1 - \hat{R} Q_0}$

ControllerDesign $\hat{Q} = \arg\min_Q \| H_{desired} - \hat{G} Q \|$

Repeat

Although we can not offer any proof at this time, we believe that this iterative procedure provides a natural approach to learning by gradually increasing the bandwidth of the controller. The essential features fall out of the fractional representation theory, in particular via the transformation from the (G, H) system in closed-loop to the (R, S)-system in open-loop, and subsequent identification of the (R, S) system to obtain estimates of (G, H).

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